Quantum mechanical probability current as electromagnetic 4-current from topological EM fields

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ABSTRACT

Starting from a complex 4-potential $A = \alpha d\beta$ we show that the 4-current density in electromagnetism and the probability current density in relativistic quantum mechanics are of identical form. With the Dirac-Clifford algebra $C\ell_{1,3}$ as mathematical basis, the given 4-potential allows topological solutions of the fields, quite similar to Bateman's construction, but with a double field solution that was overlooked previously. A more general null-vector condition is found and wave-functions of charged and neutral particles appear as topological configurations of the electromagnetic fields.

Keywords: topological fields, electromagnetism, Clifford algebra, Dirac equation, 4-current, knots.

1. MOTIVATION

The idea of topological solutions of fields to describe the internal structure of the fundamental particles of nature has been with us for quite a while.[1] The structural richness of knots on the one hand and the prospect of solutions with a finite self-energy on the other have been important motivators. Purely electromagnetic solutions have been proposed[2–4] and tried, but perhaps the main problem is the nature of the binding force that is required to ensure their stability.[5,6] Recent progress in knotted optical fields[7,8] is boosted by Bateman's construction,[9] which uses two complex scalar functions of space-time to produce all[10] null electromagnetic fields associated with the same underlying normalized Poynting field, but it has not lead to stable or electrically charged solutions. Nonetheless the reintroduction of Bateman's method may prove to be crucial in finding new, electrically charged topological solutions beyond the uncharged knots and unstable vortices[8,11–13] given so far. In this paper we will take some important steps towards charged topological solutions, first by lifting the starting point of Batemans's method to the level of the electromagnetic potentials and second by using a mathematical basis that provides both the commutation rules of the Dirac algebra and a geometric interpretation for the topological fields.

Perhaps the best motivation for this paper comes from the fundamental example of annihilation of electrons and positrons to purely electromagnetic radiation:

$$e^+ + e^- \longrightarrow \gamma + \gamma \tag{1}$$

So far there is no theory that can describe the internals of the electron and positron or the quantized nature of light, the photons produced in this simple looking reaction. No single theory can cover the details on either side of the equation, let alone any description of the intermediate evolution of the "stuff"[14] that makes this transmutation of matter into pure radiation.

Indeed, the answers that the present body of knowledge in physics has to offer about the nature of the photon and the electron.[15] is very unsatisfactory. In fact it is worse than that: self-energy divergences, non-locality of events and poor understanding of quantum spin are just a few examples (but not the least).

A short summary of the problem would be the following. First we could ask whether or not the electron does have structure? From TeV scattering experiments we know that the electron is point-like down to length scales below 10^{-18} meter. This means that, at least down to 10^{-18} meter, the Coulomb potential remains essentially intact. It also means that the electron has no internal bits, attached by springs, that can shake about and absorb energy. The electron appears to be a single object and it looks essentially like a true point, without internal structure. However the electron has a magnetic dipole, therefore, the electron cannot be a point! On top of that,

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by comparing the mass in the electron's Coulomb field to the actual mass of the electron, we find a lower limit on the electron's size of 10^{-15} meter, which is more than a 1000 times larger than the experimentally found upper limit from point-like interaction!

The electron has angular momentum $s = \frac{1}{2}\hbar$ and in a measurement one will find this value along any direction. This means that the electron cannot be described as a rigid body because then it would have only a single instantaneous axis of rotation. Hence the electron must be more like a fluid vortex. Then the electron behaves as a quantum wave or particle, it shows diffraction and its spin may be entangled...

In summary, from the mass of the electron, the scaling of the mass of the Coulomb field with electron size and the known point-like scattering of electrons, one must conclude that the electron is effectively one single object of finite size which scales with interaction energy as if it were pure electromagnetic radiation. The single decay channel of para-positronium can be interpreted as further evidence that the electron and positron are purely electromagnetic vortices of opposite angular momentum and chirality, and in a flash of optimism it would seem that a self-confined vortex of electromagnetic radiation could have all the properties required.

This is of course a bit too naive, since the nature of any stabilization or confinement mechanism is unknown. The repulsive Coulomb forces must be balanced by the so-called "Poincaré stresses" that so far have to be postulated. The hope is, however, that a proper topological theory will shed more light on this. More precisely, the stresses should perhaps be coming from some extended form of electromagnetism, from true warping of space, or a nonlinear space.[16–18]

2. OUTLINE

In this paper we use the geometric algebra or space-time-algebra[19] $C\ell_{1,3}$ as a mathematical basis in order to combine Maxwell electromagnetism and relativistic quantum mechanics. The remarkable and exceptionally useful property of the space-time algebra is that it provides, simultaneously, a proper geometrical basis for electromagnetism and a Dirac algebra to do relativistic quantum mechanics. The utility and necessity of using such a mathematical basis is obvious in the light of Eq. (1).

As a starting point the four-potential is used, but it is of a more general form, inspired by Bateman's method, and contains components from both a vector as well as a tri-vector (the dual of a vector), similar but not the same as in a Hertz-vector description.[20] This leads to a linear, first order differential theory, where Maxwell's equations take their usual form for the fields, but where sources (the 4-current) are present from scratch and do not need to be put in by hand. The sources may take the form of knotted configurations of the field. In the special case we consider, the 4-current is in fact the sum of two 4-currents and each is of identical form to a (bosonic) quantum mechanical probability current (but without appearance of Planck's constant). It appears, however, that the two currents can also be interpreted as originating from a singlet entangled spin state of a fermion and an anti-fermion (a knot and anti-knot). It follows further that the general equation of motion of the currents is a coupled wave equation for (degenerate) spin-1, spin-1/2 (Dirac equation) or spin-0 (Klein-Gordon equation) of either zero or finite mass. The potentials, fields, currents, wave functions and Dirac spinors can all be expressed in the same quantities, being the 4-gradients of two complex fields.

A very course outline of our calculations is as follows. Starting from a two complex scalar fields α and β we construct a (complex) 4-potential $A = \alpha d\beta$, and we find from dA that the fields split into two orthogonal parts $F = F_1 + F_2$, one of which has non-zero divergences. This was overlooked by others and is an important new result. Then we find that the inhomogeneous Maxwell equations $dF(\alpha, \beta) = j(\alpha, \beta)$ can be expressed in terms of α and β, with $j = j_a + j_b$ consisting of two currents and in which the current density takes the quantum mechanical form for scalar bosons $j_a = \Psi_a^* d\Psi_a - (d\Psi_a^*)\Psi_a$ or for an entangled spin state of two fermions. For this to be true, the wave function $\Psi(\alpha, \beta)$ must satisfy a coupled Dirac, Klein-Gordon or wave equation: $\Psi_b \cdot (d^2 + m_a^2)\Psi_a = \Psi_a \cdot (d^2 + m_b^2)\Psi_b$ with $(d^2 + m^2)\Psi = (id - m)(id + m)\Psi = 0$. Contrary to what is found for single knots as derived by Bateman's method,[8] the solutions found in this paper may have $\nabla \cdot \vec{E} \neq 0$ and also $\nabla \cdot \vec{B} \neq 0$, although total charge (of the two knots together) is zero.

At the level of the fields, the stage at which the solutions propagate, the theory is linear, as it should be for the general behavior of the particles and superposition of waves. There is however a second level, that of the fields squared, which is associated with the energy density of the knotted solutions. Nonlinearity is provided if we demand the fields to form a null-vector $F^2 = (F_1 + F_2)^2 = 0$. From this, an additional non-linear differential equation is found. The null-condition makes that the field vectors form a spinor; it is both a non-linear an Lorentz-invariant condition[21,22] and it makes the knot's energy-momentum Lorentz covariant. Stability of any knot solutions should come from the combination of topology and the *associated* nonlinearity[16,17] with the *implied* curved space. When combined with the foliation of the solutions when we introduce a mapping $\alpha \to \alpha^p$ and $\beta \to \beta^q$ with $p \neq q$, when p and q are > 1, this should give us the topological fields, and the particles we hope for!

The main purpose of this paper is not to prove existence or stability of detailed solutions of topological structures. We do however provide a single mathematical framework for an integrated theory of topological electromagnetism and relativistic quantum mechanics.

3. BRIEF INTRODUCTION TO THE NOTATION

In this paper we use of the real Dirac-Clifford algebra $C\ell_{1,3}$, also known the space-time algebra (STA) or geometric algebra,[19] to express electromagnetic fields and current densities as well as 4-component Dirac wave functions. Central to the Clifford algebra is the so-called geometric product of two vectors A and B, which is defined as $AB = A \cdot B + A \wedge B$ which consists of a symmetric part $A \cdot B = (AB + BA)/2$ and n anti-symmetric part $A \wedge B = (AB - BA)/2$. First we introduce the unit basis vectors e_μ, which behave exactly like the Dirac matrices γ_μ with their commutation relations $e_\mu e_\nu - e_\nu e_\mu = 2g_{\mu\nu}$ with $g_{\mu\nu} = \mathrm{diag}(+ - - -)$. By multiplication, a total of 16 so-called (unit) multivectors can be formed, for example $e_{0123} \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ which has the property that $e_{0123}^2 = -1$ and it takes the role of the unit imaginary, despite the fact that it only commutes with the even sub group $\{\mathbb{1}, e_{\mu\nu}, e_{0123}\}$. We have to give up complex numbers and it is the price we have to pay in order to keep a geometric interpretation and proper basis for Maxwell's equations on the one hand and a relativistic quantum mechanics on the other hand. The conventional Dirac algebra is the complexification of the geometric algebra $C\ell_{1,3}$. The four-potential $A = (A_0, \vec{A})$ is

$$A = e_0A_0 + e_1A_1 + e_2A_2 + e_3A_3 \equiv e_0A_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} \tag{2}$$

where we have adopted a mixed notation using 3-space vectors to separate the time- (e_0) and space-like (e_i) parts of the four vector (e_μ). This allows us to use the standard vector calculus notation and the Dirac algebra simultaneously. To many readers, this will appear to be helpful in recognizing known physics even if Clifford algebra is new to them. The column notation shows the unit basis vectors (e_i, e_j, e_k), with $i = \{i, i0, jk, 0jk\}$ (cyclic), as they project onto the unit vectors (e_x, e_y, e_z) of 3-space. The most general multivector, having all possible components, is $M = s + v + b + r + t + q$ or

$$M = s_0 + e_0v_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{v} + \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} \vec{b} + \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \vec{r} + \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} \vec{t} + e_{123}t_0 + e_{0123}q_0 \tag{3}$$

The letters refer to the nature the of the basis element: scalar, vector (polar vector), bivector (boost and rotor), trivector (pseudo vector or axial vector) and quadrivector (pseudoscalar). In our notation, it is this implicit projection that makes the connection between the Dirac matrices and the geometry of space-time. The differential operator is

$$d = e_0\partial_0 - e_1\partial_1 - e_2\partial_2 - e_3\partial_3 \tag{4}$$

The differential operating on the 4-potential is given by the geometric product $dA = d \cdot A + d \wedge A$ and this can be written explicitly as

$$dA = \partial_0A_0 + \nabla \cdot \vec{A} - \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\partial_0\vec{A} + \nabla A_0) - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \nabla \times \vec{A} \tag{5}$$

which consists of a scalar part $L = d \cdot A$ and a bivector part $F = d \wedge A$, we write $dA = L + F$. In case we put $L = 0$ we have the Lorenz gauge condition:

$$L = \partial_0 A_0 + \nabla \cdot \vec{A} = 0 \tag{6}$$

The equivalent of the Faraday or field-strength tensor $F^{\mu\nu}$ is represented by the double bivector F:

$$F = \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} \vec{E} - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \vec{B} = \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\vec{E} + e_{0123}\vec{B}) \tag{7}$$

The bivector F is also referred to as the Riemann-Silberstein vector. For a general multivector M, the hermitian conjugate is $M^\dagger = e_0 \widetilde{M} e_0$, where the tilde means reversal, for example $\widetilde{AB} = BA$ in case of two vectors.
The derivative of F yields the field differentials of Maxwell's equations, $dF = j$:

$$\begin{aligned} dF &= e_0 \nabla \cdot \vec{E} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \left(\nabla \times \vec{B} - \partial_0 \vec{E}\right) + e_{123} \nabla \cdot \vec{B} - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} \left(\nabla \times \vec{E} + \partial_0 \vec{B}\right) \\ &= e_0 j_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{j} = j \end{aligned} \tag{8}$$

where j is the 4-current density, the source term which must be put in by hand. It is also the part that is associated with the self-energy problem of electrical (point) charge. It is the bold purpose of this paper to find a detailed mathematical description of the structure of charged particles (and their motion) as knotted solutions of fields, potentials, wave functions and energy flow. We will show that we can write both sides of Maxwell's equations in terms of two complex scalar fields α and β: $dF(\alpha, \beta) = j(\alpha, \beta)$. In doing so, j may take the form of a quantum probability 4-current with wave function $\Psi(\alpha, \beta)$ that obeys relativistic quantum mechanics.

4. AN APPROACH THAT ALLOWS TOPOLOGICAL SOLUTIONS

To build electromagnetism, the conventional starting point is the four variables of the four-potential $A = (A_0, \vec{A})$. Another approach is to start from another set of four variables, making up two complex numbers α and β according to the method of Bateman.[7–10] Here, these are projected onto the space-time algebra as follows:

$$\alpha = \mathbb{1}\alpha_s + e_{0123}\alpha_q \tag{9}$$

$$\beta = \mathbb{1}\beta_s + e_{0123}\beta_q \tag{10}$$

This particular choice seems to be natural because the quadrivector e_{0123} commutes with the even algebra $\{\mathbb{1}, e_{0i}, e_{jk}, e_{0123}\}$ and thus behaves as the imaginary number $\mathbf{i}$ with respect to the electromagnetic fields. The complex conjugate of α is $\alpha^* \equiv \alpha_s - e_{0123}\alpha_q$, the same as the hermitian conjugate $\alpha^\dagger \equiv e_0 \widetilde{\alpha} e_0 = \alpha_s - e_{0123}\alpha_q$, as should be for a scalar. The 4-gradient of α is

$$d\alpha = d\alpha_s - e_{0123} d\alpha_q = e_0 \partial_0 \alpha_s + e_{123} \partial_0 \alpha_q - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \nabla \alpha_s - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} \nabla \alpha_q \tag{11}$$

Note that $d\alpha^* \equiv (d\alpha)^\# \neq (d\alpha)^\dagger$, but $e_0 d\alpha^* e_0 = (d\alpha)^\dagger = d^\dagger \alpha$.
Similar to what was done in Section 3, we will now derive electromagnetism from a four potential $A = \alpha d\beta$ that has four vector components (as usual), but also four trivector components. The total of eight components stem from only four numbers (two complex numbers) so that there is exactly the same amount of input as usual, but distributed differently:

$$A = \alpha d\beta = e_0 \alpha^* \partial_0 \beta - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \alpha^* \nabla \beta \tag{12}$$

contains α^*, not α. The differential dA is of the following structure:

$$dA = d(\alpha d\beta) = (d\alpha)d\beta + \dot{d}\alpha\dot{d}\beta = (d\alpha)d\beta + \alpha^* d^2\beta \tag{13}$$

where we applied Leibnitz' rule $d(FG) = \dot{d}\dot{F}G + \dot{d}F\dot{G} = (dF)G + \dot{d}F\dot{G} = e_{dFG}[(dF)G + F(dG)]$, but adapted to the non-commuting algebra in which the overdot labels the subject of derivation. We find that dA gives a term $\alpha^* d^2\beta$ containing both a scalar and pseudoscalar part as well as a product $(d\alpha)d\beta$. As we will see in a moment, the latter contains exactly all of the (potentially topological) fields by Bateman. The product $(d\alpha)d\beta$ is

$$\begin{aligned}
(d\alpha)d\beta &= \partial_0\alpha_s\partial_0\beta_s + \partial_0\alpha_q\partial_0\beta_q - \nabla\alpha_s\cdot\nabla\beta_s - \nabla\alpha_q\cdot\nabla\beta_q \\
&+ e_{0123}(\partial_0\alpha_s\partial_0\beta_q - \partial_0\alpha_q\partial_0\beta_s - \nabla\alpha_s\cdot\nabla\beta_q + \nabla\alpha_q\cdot\nabla\beta_s) \\
&+ \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\nabla\alpha_s\times\nabla\beta_q - \nabla\alpha_q\times\nabla\beta_s + \partial_0\alpha_s\nabla\beta_s + \partial_0\alpha_q\nabla\beta_q - \partial_0\beta_s\nabla\alpha_s - \partial_0\beta_q\nabla\alpha_q) \\
&+ \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} (\nabla\alpha_s\times\nabla\beta_s + \nabla\alpha_q\times\nabla\beta_q + \partial_0\alpha_q\nabla\beta_s - \partial_0\alpha_s\nabla\beta_q + \partial_0\beta_q\nabla\alpha_s - \partial_0\beta_s\nabla\alpha_q)
\end{aligned} \tag{14}$$

which consists of a scalar part S_0, a pseudoscalar part Q_0 and a bivector part $F = d\alpha\wedge d\beta$, $(d\alpha)d\beta = S_0 + F + Q_0$. An extra scalar S_A and pseudoscalar Q_A come from:

$$\alpha^* d^2\beta = \alpha_s\partial_0^2\beta_s + \alpha_q\partial_0^2\beta_q - \alpha_s\nabla^2\beta_s - \alpha_q\nabla^2\beta_q + e_{0123}(\alpha_s\partial_0^2\beta_q - \alpha_q\partial_0^2\beta_s - \alpha_s\nabla^2\beta_q + \alpha_q\nabla^2\beta_s) \tag{15}$$

We define $\Upsilon_s = S_0 + S_A$ and $e_{0123}\Upsilon_q = Q_0 + Q_A$ and find

$$\Upsilon_s = d\alpha_s\cdot d\beta_s + d\alpha_q\cdot d\beta_q + \alpha_s d^2\beta_s + \alpha_q d^2\beta_q \tag{16}$$

$$\Upsilon_q = d\alpha_s\cdot d\beta_q - d\alpha_q\cdot d\beta_s + \alpha_s d^2\beta_q - \alpha_q d^2\beta_s \tag{17}$$

we may (but won't) set the Lorenz gauge condition $\Upsilon_s \equiv L = 0$. At this point it is not yet clear what gauge should be set. Recently, it has been proposed to add to Υ another real or complex scalar field P, named "p-vot", that may act as a confinement source term for localized field solutions,[23] but such extra assumptions may not be required and can, in fact, ruin the whole theory. Other conditions may have to be fulfilled first and take away some freedom of choice. Normally, when we start from a 4-vector potential, we would have $\Upsilon_q \equiv 0$, but now we have this pseudoscalar field at our disposal, the particularly intriguing possibility unfolds to form another complex field $\Upsilon = \Upsilon_s + e_{0123}\Upsilon_q = \alpha^* d^2\beta + d\alpha\cdot d\beta$ which may be set to obey the massless Dirac (or neutrino) equation $d\Upsilon = 0$. Of course, Υ must then take a bispinor form and this may only make sense once we apply a map $\alpha \leftrightarrow \alpha^p$, $\beta \leftrightarrow \beta^q$ with $p \neq q$ that introduces a double covering solution. A question is then whether topological solutions using Eqs. (16) and (17) may indeed produce spin 1/2 and perhaps also show some clue to what the weak interaction may be within the context of this mathematical framework. Further, note that we can write $dA = \Upsilon + F = \alpha^* d^2\beta + d\alpha\cdot d\beta + d\alpha\wedge d\beta$.

The Riemann-Silberstein vector appears to split naturally into two parts $F = F_1 + F_2$:

$$\begin{aligned}
F_1 &= \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\nabla\alpha_s\times\nabla\beta_q - \nabla\alpha_q\times\nabla\beta_s) \\
&+ \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} (\nabla\alpha_s\times\nabla\beta_s + \nabla\alpha_q\times\nabla\beta_q)
\end{aligned} \tag{18}$$

$$\begin{aligned}
F_2 &= \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\partial_0\alpha_s\nabla\beta_s + \partial_0\alpha_q\nabla\beta_q - \partial_0\beta_s\nabla\alpha_s - \partial_0\beta_q\nabla\alpha_q) \\
&+ \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} (\partial_0\alpha_q\nabla\beta_s - \partial_0\alpha_s\nabla\beta_q + \partial_0\beta_q\nabla\alpha_s - \partial_0\beta_s\nabla\alpha_q)
\end{aligned} \tag{19}$$

To compare our results to the literature we use the general form Eq. (7) and a mixed notation (care should be taken!) using $\alpha^* = \alpha_s - e_{0123}\alpha_q$ and $\beta = \beta_s + e_{0123}\beta_q$:

$$F_1 = \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \nabla\alpha^* \times \nabla\beta = -e_{0123} \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} \nabla\alpha^* \times \nabla\beta \tag{20}$$

$$F_2 = \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\partial_0\alpha^*\nabla\beta - \partial_0\beta\nabla\alpha^*) \tag{21}$$

It appears that the $F_1 \cdot F_2 = 0$ (hence $F_1 \perp F_2$), because both $\nabla\beta \cdot (\nabla\alpha^* \times \nabla\beta) = 0$ and $\nabla\alpha^* \cdot (\nabla\alpha^* \times \nabla\beta) = 0$. Note however that in the special case $F = F_1 = F_2$, we find

$$\nabla\alpha^* \times \nabla\beta = e_{0123}(\partial_0\alpha^*\nabla\beta - \partial_0\beta\nabla\alpha^*) \tag{22}$$

which may seem puzzling since we have also just found that $F_1 \perp F_2$, but one should keep in mind that there are more types of orthogonality, it can be of either a geometric (vector orientation) or differential (vector calculus) nature. In the case of electromagnetic radiation we always find that F is a so-called "null vector": the projection on to itself has zero length, $F^2 = E^2 - B^2 + 2e_{0123}\vec{E} \cdot \vec{B} = 0$, but it does not at all mean that the vector vanishes. Instead it means that the different parts of F^2 are Lorentz invariant scalars.

The form of $F = F_1 = F_2$ in Eq. (22) is precisely what is used in Ref. 8, but with $\alpha \leftrightarrow \alpha^*$, $\beta \leftrightarrow \beta$ and $\mathbf{i} \leftrightarrow e_{0123}$ such that $\nabla\alpha \times \nabla\beta = \mathbf{i}(\partial_0\alpha\nabla\beta - \partial_0\beta\nabla\alpha)$. To generate topological solutions, a map should be made $\alpha \leftrightarrow \alpha^p$, $\beta \leftrightarrow \beta^q$ with $p \neq q$ so that closed field lines get linked. Clearly, Ref. 8 demands that both F_1 and F_2 are null. Their starting point was Bateman's method with $\nabla\alpha \times \nabla\beta$ which represents only a subset of our starting point $(d\alpha)d\beta$. The essential difference and important insight gained in this paper is that our starting point offers us an opportunity to form null vectors F, even if the F_1 and F_2 themselves are not null, while they do fulfill the orthogonality condition $F_1 \cdot F_2 = 0$ so that in our case $F^2 = (F_1 + F_2)^2 = F_1^2 + F_2^2 = 0$. We want $F_1 \neq F_2$, not $F_1 = F_2$, this becomes quite clear if we derive Maxwell's equations: $dF = d(F_1 + F_2)$. It follows straight away that due to general identities in vector calculus

$$\nabla \cdot \vec{E}_1 = 0 \tag{23}$$

$$\nabla \cdot \vec{B}_1 = 0 \tag{24}$$

$$\nabla \times \vec{B}_2 = \partial_0(\nabla\alpha_s \times \nabla\beta_q) - \partial_0(\nabla\alpha_q \times \nabla\beta_s) = \partial_0\vec{E}_1 \tag{25}$$

$$\nabla \times \vec{E}_2 = \partial_0(\nabla\alpha_s \times \nabla\beta_s) + \partial_0(\nabla\alpha_q \times \nabla\beta_q) = -\partial_0\vec{B}_1 \tag{26}$$

so that

$$\begin{aligned} dF = d(F_1 + F_2) &= e_0\nabla \cdot \vec{E}_2 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \left(\nabla \times \vec{B}_1 - \partial_0\vec{E}_2\right) \\ &+ e_{123}\nabla \cdot \vec{B}_2 - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} \left(\nabla \times \vec{E}_1 + \partial_0\vec{B}_2\right) \end{aligned} \tag{27}$$

This is a quite remarkable result. The role of F_1 and F_2 appear to be very different. In case we demand, as in the literature,[8] that $F = F_1 = F_2$, the divergences will vanish. Not good for a topological, or any description of a charged particle. We will simply use the full expression $F = F_1 + F_2$. The Eqs. (25) and (26) couple the fields between the different Riemann-Silberstein vectors, and may be looked at as providing an interaction mechanism.

Interestingly we can make another split of the fields $F = F_m + F_e$ for which F_m depends only on $\vec{E}_1$ and $\vec{B}_2$ and has $j = 0$ and F_e depends only on $\vec{E}_2$ and $\vec{B}_1$ and has $j^m = 0$.

$$dF_m(E_1, B_2) = j^m \tag{28}$$

$$dF_e(E_2, B_1) = j \tag{29}$$

The latter is of the form we see in the natural world, where no clear indication for the existence of magnetic monopoles is found. We may speculate that a magnetic monopole with F_m lives inside particles and with F_e on the outside to make them electrically charged. By the Dirac monopole construction,[24] this would automatically imply that the particle must have quantized charge as well as having spin 1/2. It should be mentioned at this point that the author is well aware that, despite the purpose of this paper, hiding a monopole is just as impossible and ludicrous as producing a charge from fields, let alone making light go round in circles, all by itself!. Nonetheless, we will keep pushing, if only to see what in the end would be required, both mathematically and in our thinking of space-time, to realize what nature is already doing. Stability of any knotted solutions as well as the divergence of fields should, some how, come from the combination of topology and the *associated* nonlinearity[16,17] with the *implied* curved space. It is our task to show exactly how this is (im)possible.

Suppose we will succeed in making topological configurations of the electromagnetic fields, without charges. Normally we see fields act on charges, but what if we manage to replace the charges merely by just the right fields? Some readers may wonder whether fields may act on fields. There is a way to prove, by Hamilton's principle of virtual work, that indeed fields do act on fields, without violating the superposition principle.[25]

5. THE 4-CURRENT

In this section we want to make a connection between the electromagnetic 4-current and the probability current in quantum mechanics as, incidentally, is done already (for fermions) in quantum electrodynamics:

$$\partial_\nu F^{\nu\mu} = e\bar{\psi}\gamma^\mu\psi = j^\mu \tag{30}$$

However, what we want is to obtain a generalized form of this for both fermions and bosons, perhaps as a result of the topology of the underlying continuous (electromagnetic) fields. In this paper the role of our start potential $A = \alpha d\beta$ will appear to be twofold: First, it will allow us to write the electromagnetic 4-current density in a similar form as the quantum mechanical 4-current density. Second, its development, continuity and boundary conditions will provide (linear, nonlinear and topological) equations and a template for the structure of the associated particles. In terms of our insights, the physical interpretation of Eq. (30) would be that all electromagnetism and all quantum mechanics as well as the stuff particles and waves are made from are given by the complex fields α and β. It should be noted that the exact form of the complex fields α and β is still open, and may provide a whole spectrum of solutions. Also it should be noted that at this point we do not have any indication nor pretension that we may solve the nature of the quantized action $\hbar$ or the charge e, but having said so, looking at the literature, we learn that it is also a possibility that we will find answers for some fundamental constants.[11,12,26] Other relevant literature is found where gradient sources have been tried.[27,28]

In this section we will show something remarkable, namely that (a specific form of) the inhomogeneous Maxwell equations $dF = j$ (that means the fields AND the sources) follow from our starting assumption $A = \alpha d\beta$. We will find that the electromagnetic current has exactly the same form as the quantum mechanical expression of TWO probability currents: $j = j_a + j_b$, each of which can be written as a bosonic current $j_a = \Psi_a^* d\Psi_a - \Psi_a d\Psi_a^*$. This is a new result. Both F and Ψ (and hence j) can be expressed in terms of the 4-gradients of two complex fields $d\alpha$ and $d\beta$. A current density j_a of this quantum mechanical form can only follow from some non-abelian mathematics and not from Maxwell's equations when written in the usual way and derived from a 4-vector potential only. It should be noted that, as a consequence of our starting point, the magnetic divergence does not automatically vanish and that we must deal, in our theory, with the experimental fact that magnetic monopoles are not observed.

The inhomogeneous Maxwell's equations (7) with our specific starting potential $A = \alpha d\beta$, leads to Eqs. (23)-(27), and the currents are defined as follows:

$$\nabla \cdot \vec{E}_2 = \rho = j_0 \tag{31}$$

$$\nabla \times \vec{B}_1 - \partial_0 \vec{E}_2 = \vec{j} \tag{32}$$

$$\nabla \cdot \vec{B}_2 = j_0^m \ \ (=0) \tag{33}$$

$$\nabla \times \vec{E}_1 + \partial_0 \vec{B}_2 = -\vec{j^m} \ \ (=0) \tag{34}$$

where the last two terms are zero in the absence of magnetic monopoles. If we take the expressions Eqs. (18) and (19), after some rearrangements of terms, we can write the electric charge and current density in terms of $d\alpha$ and $d\beta$:

$$\begin{aligned}\rho &= \nabla\beta_s\cdot\partial_0\nabla\alpha_s + \nabla\beta_q\cdot\partial_0\nabla\alpha_q - \nabla\alpha_s\cdot\partial_0\nabla\beta_s - \nabla\alpha_q\cdot\partial_0\nabla\beta_q \\ &- \partial_0\beta_s\nabla^2\alpha_s - \partial_0\beta_q\nabla^2\alpha_q + \partial_0\alpha_s\nabla^2\beta_s + \partial_0\alpha_q\nabla^2\beta_q \end{aligned} \tag{35}$$

$$\begin{aligned}\vec{j} &= \partial_0\beta_s\nabla\partial_0\alpha_s + \partial_0\beta_q\nabla\partial_0\alpha_q - \partial_0\alpha_s\nabla\partial_0\beta_s - \partial_0\alpha_q\nabla\partial_0\beta_q \\ &- (\nabla\beta_s\cdot\nabla)\nabla\alpha_s - (\nabla\beta_q\cdot\nabla)\nabla\alpha_q + (\nabla\alpha_s\cdot\nabla)\nabla\beta_s + (\nabla\alpha_q\cdot\nabla)\nabla\beta_q \\ &- \nabla\beta_s(\partial_0^2\alpha_s - \nabla^2\alpha_s) - \nabla\beta_q(\partial_0^2\alpha_q - \nabla^2\alpha_q) + \nabla\alpha_s(\partial_0^2\beta_s - \nabla^2\beta_s) + \nabla\alpha_q(\partial_0^2\beta_q - \nabla^2\beta_q)\end{aligned} \tag{36}$$

Using the expressions above, we may try to make a connection between the charge density in Maxwell electromagnetism and the probability charge and current density for scalar bosons in quantum mechanics:

$$\rho_{qm} = \frac{i\hbar}{2m}\left(\psi^*\partial_0\psi - \psi\partial_0\psi^*\right) \tag{37}$$

$$\vec{j}_{qm} = -\frac{i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{38}$$

Comparing the two sets of two equations above, we find that the first line of both ρ and $\vec{j}$ are looking good, and also that the last lines of both ρ and $\vec{j}$ can be fixed. We will want to use a set of what looks like some coupled wave equations, for all components, similar to the last line of $\vec{j}$ to exchange the second order space derivatives with those of time (the d'Alembertian $\Box = d^2 = \partial_0^2 - \nabla^2$). In the expression for the charge density ρ we try:

$$0 = \partial_0\beta_s(\partial_0^2\alpha_s - \nabla^2\alpha_s) + \partial_0\beta_q(\partial_0^2\alpha_q - \nabla^2\alpha_q) - \partial_0\alpha_s(\partial_0^2\beta_s - \nabla^2\beta_s) - \partial_0\alpha_q(\partial_0^2\beta_q - \nabla^2\beta_q) \tag{39}$$

and combined with last line of $\vec{j}$, this demands a part of 4-current to vanish:

$$j^e_{zero} = d\beta_s d^2\alpha_s - d\alpha_s d^2\beta_s + d\beta_q d^2\alpha_q - d\alpha_q d^2\beta_q = 0 \tag{40}$$

and doing all this similar for $\vec{j}^m$, another coupled wave equation is found:

$$j^m_{zero} = d\beta_q d^2\alpha_s - d\alpha_s d^2\beta_q - d\beta_s d^2\alpha_q + d\alpha_q d^2\beta_s = 0 \tag{41}$$

As will become clear in the next page or two, the Eqs. (40) and (41) are necessary conditions in order to find a form of the current density that resembles that of quantum mechanics, and it may have consequences for the consistency or validity of our approach. We will come back to this to show the dramatic implications imposed. First, we concentrate on the form for ρ and $\vec{j}$ that is now implied:

$$\begin{aligned}\rho &= \partial_0\alpha_s\partial_0^2\beta_s - \partial_0\beta_s\partial_0^2\alpha_s + \partial_0\alpha_q\partial_0^2\beta_q - \partial_0\beta_q\partial_0^2\alpha_q \\ &- \nabla\alpha_s\cdot\partial_0\nabla\beta_s + \nabla\beta_s\cdot\partial_0\nabla\alpha_s - \nabla\alpha_q\cdot\partial_0\nabla\beta_q + \nabla\beta_q\cdot\partial_0\nabla\alpha_q\end{aligned} \tag{42}$$

$$\begin{aligned}\vec{j} &= -\partial_0\alpha_s\nabla\partial_0\beta_s + \partial_0\beta_s\nabla\partial_0\alpha_s - \partial_0\alpha_q\nabla\partial_0\beta_q + \partial_0\beta_q\nabla\partial_0\alpha_q \\ &+ (\nabla\alpha_s\cdot\nabla)\nabla\beta_s - (\nabla\beta_s\cdot\nabla)\nabla\alpha_s + (\nabla\alpha_q\cdot\nabla)\nabla\beta_q - (\nabla\beta_q\cdot\nabla)\nabla\alpha_q\end{aligned} \tag{43}$$

In the standard notation of Eqs. (37) and (38), the wave function ψ may be, in the rest frame, a complex scalar describing a particle without spin, a spin-0 boson. Our wave function is complex with four-components $\Psi = e_0\Psi_{r0} + e_1\Psi_{r1} + e_2\Psi_{r2} + e_3\Psi_{r3} + e_{0123}(e_0\Psi_{i0} + e_1\Psi_{i1} + e_2\Psi_{i2} + e_3\Psi_{i3})$, and can be regarded equally well as a vector or a spinor, and may have enough degrees of freedom to carry a wave function of higher spin. We would expect something of the form

$$j_\mu = \frac{i\hbar}{2m}\left(\Psi^*\cdot\partial_\mu\Psi - (\partial_\mu\Psi^*)\cdot\Psi\right) \approx \frac{\hbar}{m}\left(\Psi_i\cdot\partial_\mu\Psi_r - \Psi_r\cdot\partial_\mu\Psi_i\right) \tag{44}$$

In the first instance we may want to match the proper set of vectors Ψ and derivatives $d\Psi$ without bothering too much with the (non)-commuting nature of our unit imaginary pseudoscalar e_{0123}, hence the "approximately equal" sign in Eq. (44). We may amongst others be able to choose an appropriate conjugate of Ψ to deal with the issue. On closer inspection, we may find that Ψ is only scalar after all, or perhaps an anti-symmetric tensor describing a (degenerate) spin-1 particle. We may write some wave function that looks as follows:

$$\begin{aligned} \Psi &= d\alpha + e_{0123}d\beta = \Psi_r - e_{0123}\Psi_i \\ &= e_0(\partial_0\alpha_s + \partial_0\beta_q + e_{0123}(\partial_0\alpha_q - \partial_0\beta_s)) - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (\nabla\alpha_s + \nabla\beta_q + e_{0123}(\nabla\alpha_q - \nabla\beta_s)) \end{aligned} \tag{45}$$

so that $\Psi_r = d\alpha_s + d\beta_q$ and $\Psi_i = d\alpha_q - d\beta_s$ and the charge density takes the form

$$\begin{aligned} \rho_\Psi &= \Psi_i \cdot \partial_0\Psi_r - \Psi_r \cdot \partial_0\Psi_i = (d\alpha_q - d\beta_s)\cdot\partial_0(d\alpha_s + d\beta_q) - (d\alpha_s + d\beta_q)\cdot\partial_0(d\alpha_q - d\beta_s) \\ &= \partial_0\alpha_s\partial_0^2\beta_s - \partial_0\beta_s\partial_0^2\alpha_s + \partial_0\alpha_q\partial_0^2\beta_q - \partial_0\beta_q\partial_0^2\alpha_q \\ &- \partial_0\alpha_s\partial_0^2\alpha_q + \partial_0\alpha_q\partial_0^2\alpha_s - \partial_0\beta_s\partial_0^2\beta_q + \partial_0\beta_q\partial_0^2\beta_s \\ &- \nabla\alpha_s\cdot\partial_0\nabla\beta_s + \nabla\beta_s\cdot\partial_0\nabla\alpha_s - \nabla\alpha_q\cdot\partial_0\nabla\beta_q + \nabla\beta_q\cdot\partial_0\nabla\alpha_q \\ &+ \nabla\alpha_s\cdot\partial_0\nabla\alpha_q - \nabla\alpha_q\cdot\partial_0\nabla\alpha_s + \nabla\beta_s\cdot\partial_0\nabla\beta_q - \nabla\beta_q\cdot\partial_0\nabla\beta_s \end{aligned} \tag{46}$$

This can be brought into the same form as Eq. (42) by introducing a conjugate wave function $\Psi^\# = d\alpha - e_{0123}d\beta$ that, incidentally, corresponds to $F^\# = -F$ so that

$$\begin{aligned} \frac{1}{2}(\rho_\Psi - \rho_{\Psi^\#}) &= \partial_0\alpha_s\partial_0^2\beta_s - \partial_0\beta_s\partial_0^2\alpha_s + \partial_0\alpha_q\partial_0^2\beta_q - \partial_0\beta_q\partial_0^2\alpha_q \\ &- \nabla\alpha_s\cdot\partial_0\nabla\beta_s + \nabla\beta_s\cdot\partial_0\nabla\alpha_s - \nabla\alpha_q\cdot\partial_0\nabla\beta_q + \nabla\beta_q\cdot\partial_0\nabla\alpha_q \end{aligned} \tag{47}$$

which would describe some state of a pair of particles of opposite charge. What we see in Eqs. (42) and (43) is a sandwich of ∂_0 between the 4-gradients of the complex fields α and β, making sixteen components, and this implies that we are not dealing with a single current, but a double current. Simply speaking we have two scalar particles of opposite charge. However, the situation could be more interesting and we may be dealing with two oppositely charged particles in an entangled state of zero total spin. Indeed, then the particles may even be of fermionic nature. We seem to have enough degrees of freedom between the α's and β's to support this and to produce a wave function with non-trivial topology, but that is for later. It is possible to produce an alternatively set of wave functions:

$$\mathbf{u} = d\alpha_s + e_{0123}d\beta_s = \mathbf{u}_r + e_{0123}\mathbf{u}_i \tag{48}$$

$$\mathbf{v} = d\alpha_q + e_{0123}d\beta_q = \mathbf{v}_r + e_{0123}\mathbf{v}_i \tag{49}$$

so that $\Psi = d\alpha + e_{0123}d\beta = \mathbf{u} - e_{0123}\mathbf{v} = (\mathbf{u}_r + \mathbf{v}_i) + e_{0123}(\mathbf{u}_i - \mathbf{v}_r)$, $\Psi^* = d\alpha^* - e_{0123}d\beta^* = \mathbf{u}^\# + e_{0123}\mathbf{v}^\# = (\mathbf{u}_r + \mathbf{v}_i) - e_{0123}(\mathbf{u}_i - \mathbf{v}_r)$ and $\Psi^\# = d\alpha - e_{0123}d\beta = \mathbf{u}^\# - e_{0123}\mathbf{v}^\# = (\mathbf{u}_r - \mathbf{v}_i) - e_{0123}(\mathbf{u}_i + \mathbf{v}_r)$ and further $\Psi^{*\#} = \Psi^{\#*} = d\alpha^* + e_{0123}d\beta^* = \mathbf{u} + e_{0123}\mathbf{v} = (\mathbf{u}_r - \mathbf{v}_i) + e_{0123}(\mathbf{u}_i + \mathbf{v}_r)$ and

$$\rho = (\rho_\Psi - \rho_{\Psi^\#})/2 = -(\rho_u + \rho_v)/2 \tag{50}$$

We have digressed a bit from our purpose and must bring the current density of Eq. (43) in a matching 4-vector form. It appears to have an awkward dot product with the gradient. Fortunately, the vector it operates on is itself is a 4-gradient of a complex field so that the order of derivatives can be swapped to give us the expression that we recognize. It can be seen as follows, by looking at $\vec{j}$ component wise:

$$\begin{aligned} j_i &= \partial_0\beta_s\partial_i\partial_0\alpha_s - \partial_0\alpha_s\partial_i\partial_0\beta_s + \partial_0\beta_q\partial_i\partial_0\alpha_q - \partial_0\alpha_q\partial_i\partial_0\beta_q \\ &+ (\partial_1\alpha_s\partial_1 + \partial_2\alpha_s\partial_2 + \partial_3\alpha_s\partial_3)\partial_i\beta_s - (\partial_1\beta_s\partial_1 + \partial_2\beta_s\partial_2 + \partial_3\beta_s\partial_3)\partial_i\alpha_s \\ &+ (\partial_1\alpha_q\partial_1 + \partial_2\alpha_q\partial_2 + \partial_3\alpha_q\partial_3)\partial_i\beta_q - (\partial_1\beta_q\partial_1 + \partial_2\beta_q\partial_2 + \partial_3\beta_q\partial_3)\partial_i\alpha_q \\ &= d\beta_s\cdot\partial_i d\alpha_s - d\alpha_s\cdot\partial_i d\beta_s + d\beta_q\cdot\partial_i d\alpha_q - d\alpha_q\cdot\partial_i d\beta_q \qquad (51) \\ &= (d\beta_s\cdot d)\partial_i\alpha_s - (d\alpha_s\cdot d)\partial_i\beta_s + (d\beta_q\cdot d)\partial_i\alpha_q - (d\alpha_q\cdot d)\partial_i\beta_q \qquad (52) \end{aligned}$$

Indeed, this shows the right kind of form to match the charge and current density, and our initial choice of wave function was close enough. If we sort this and also include j_0 then for all the j_μ we have:

$$\begin{aligned} j &= (d\alpha_s \cdot d)d\beta_s - (d\beta_s \cdot d)d\alpha_s + (d\alpha_q \cdot d)d\beta_q - (d\beta_q \cdot d)d\alpha_q \\ &= (\mathbf{u}_r \cdot d)\mathbf{u}_i - (\mathbf{u}_i \cdot d)\mathbf{u}_r + (\mathbf{v}_r \cdot d)\mathbf{v}_i - (\mathbf{v}_i \cdot d)\mathbf{v}_r \\ &= \frac{e_{0123}}{2}[(\mathbf{u}^\# \cdot d)\mathbf{u} - (\mathbf{u} \cdot d)\mathbf{u}^\# + (\mathbf{v}^\# \cdot d)\mathbf{v} - (\mathbf{v} \cdot d)\mathbf{v}^\#] \end{aligned} \tag{53}$$

so that $j = -(j_u + j_v)/2$

Note that to obtain this result, it is required that the vectors $\mathbf{u}$ and $\mathbf{v}$ are 4-gradients to begin with and that this also seems to eliminate the possibility of producing an anti-symmetric tensor $\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu$ to describe a spin-1 boson. In summary, we have found that the 4-current density reads

$$j_\Psi - j_{\Psi^\#} = (\Psi^* \cdot d)\Psi - (\Psi \cdot d)\Psi^* - (\Psi^{\#*} \cdot d)\Psi^\# - (\Psi^\# \cdot d)\Psi^{\#*}) \tag{54}$$

which may describe an (in)coherent state of a particle pair of conjugate charge and/or spin. To obtain this result, however, we had to fulfill some coupled wave equations, Eqs. (40) and (41), which we will investigate next.

6. A SHORTCUT

As a shortcut to the results obtained so far, using $d\alpha = \mathbf{u}_r - e_{0123}\mathbf{v}_r$ and $d\beta = \mathbf{u}_i - e_{0123}\mathbf{v}_i$, we can write

$$A = \alpha d\beta \tag{55}$$

$$dA = \Upsilon + F = \alpha^* d^2\beta + d\alpha \cdot d\beta + d\alpha \wedge d\beta \tag{56}$$

$$\Upsilon = d \cdot A = \alpha^* d^2\beta + d\alpha \cdot d\beta \tag{57}$$

$$\begin{aligned} F &= d \wedge A = d\alpha \wedge d\beta = (\mathbf{u}_r - e_{0123}\mathbf{v}_r) \wedge (\mathbf{u}_i - e_{0123}\mathbf{v}_i) \\ &= \mathbf{u}_r \wedge \mathbf{u}_i + \mathbf{v}_r \wedge \mathbf{v}_i + e_{0123}(\mathbf{u}_r \wedge \mathbf{v}_i - \mathbf{v}_r \wedge \mathbf{u}_i) \end{aligned} \tag{58}$$

$$dF = d(d\alpha \wedge d\beta) \tag{59}$$

$$\begin{aligned} &= (\mathbf{u}_r \cdot d)\mathbf{u}_i - (\mathbf{u}_i \cdot d)\mathbf{u}_r + (\mathbf{v}_r \cdot d)\mathbf{v}_i - (\mathbf{v}_i \cdot d)\mathbf{v}_r \\ &+ \mathbf{u}_i d \cdot \mathbf{u}_r - \mathbf{u}_r d \cdot \mathbf{u}_i + \mathbf{v}_i d \cdot \mathbf{v}_r - \mathbf{v}_r d \cdot \mathbf{v}_i \\ &+ e_{0123}[(\mathbf{u}_r \cdot d)\mathbf{v}_i - (\mathbf{v}_i \cdot d)\mathbf{u}_r - (\mathbf{v}_r \cdot d)\mathbf{u}_i + (\mathbf{u}_i \cdot d)\mathbf{v}_r \\ &+ \mathbf{v}_i d \cdot \mathbf{u}_r - \mathbf{u}_r d \cdot \mathbf{v}_i - \mathbf{u}_i d \cdot \mathbf{v}_r + \mathbf{v}_r d \cdot \mathbf{u}_i] \end{aligned} \tag{60}$$

where we have used that $\mathbf{u}_r$, $\mathbf{u}_i$, $\mathbf{v}_r$ and $\mathbf{v}_i$ are all gradient vectors, in combination with the following identities:

$$d(\mathbf{u}_r \wedge \mathbf{u}_i) \equiv (\mathbf{u}_r \cdot d)\mathbf{u}_i - (\mathbf{u}_i \cdot d)\mathbf{u}_r + \mathbf{u}_i d \cdot \mathbf{u}_r - \mathbf{u}_r d \cdot \mathbf{u}_i \tag{61}$$

$$d \wedge \mathbf{u}_r = d \wedge d\alpha_s \equiv 0 \tag{62}$$

now we demand

$$0 = \mathbf{u}_i d \cdot \mathbf{u}_r - \mathbf{u}_r d \cdot \mathbf{u}_i + \mathbf{v}_i d \cdot \mathbf{v}_r - \mathbf{v}_r d \cdot \mathbf{v}_i \tag{63}$$

$$0 = \mathbf{v}_i d \cdot \mathbf{u}_r - \mathbf{u}_r d \cdot \mathbf{v}_i - \mathbf{u}_i d \cdot \mathbf{v}_r + \mathbf{v}_r d \cdot \mathbf{u}_i \tag{64}$$

and identify $dF = j + j^m$, what is exactly of the quantum mechanical form, except for the fact that there are two currents j_u and j_v for both j and j^m related to the two wave functions $\mathbf{u}$ and $\mathbf{v}$.

$$j = (\mathbf{u}_r \cdot d)\mathbf{u}_i - (\mathbf{u}_i \cdot d)\mathbf{u}_r + (\mathbf{v}_r \cdot d)\mathbf{v}_i - (\mathbf{v}_i \cdot d)\mathbf{v}_r \tag{65}$$

$$j^m = (\mathbf{u}_r \cdot d)\mathbf{v}_i - (\mathbf{v}_i \cdot d)\mathbf{u}_r - (\mathbf{v}_r \cdot d)\mathbf{u}_i + (\mathbf{u}_i \cdot d)\mathbf{v}_r \tag{66}$$

As has been said before, “The same equations have the same solutions”, and we must realize that we have a double (quantum mechanical) current, of bosons, but of purely electromagnetic origin. It should be expected that the physical interpretation of these mathematical results will require most of our attention.

7. ON THE COUPLED WAVE EQUATIONS

Here we come back to the necessary conditions from Eqs. (40) and (41), or Eqs. (63) and (64):

$$j^e_{zero} = d\beta_s d^2\alpha_s - d\alpha_s d^2\beta_s + d\beta_q d^2\alpha_q - d\alpha_q d^2\beta_q = 0 \tag{67}$$
$$j^m_{zero} = d\beta_q d^2\alpha_s - d\alpha_s d^2\beta_q - d\beta_s d^2\alpha_q + d\alpha_q d^2\beta_s = 0 \tag{68}$$

First we should realize that the equations can be split, since they have two independent parts that each may represent a constant, non-zero, but opposite charge and current.

$$d\beta_s d^2\alpha_s - d\alpha_s d^2\beta_s = j^e_z = d\alpha_q d^2\beta_q - d\beta_q d^2\alpha_q \tag{69}$$
$$d\beta_q d^2\alpha_s - d\alpha_s d^2\beta_q = j^m_z = d\beta_s d^2\alpha_q - d\alpha_q d^2\beta_s \tag{70}$$

At this point we should say a lot more about the possible occurrence of magnetic monopoles and the possibility of charge quantization from Eqs. (69) and (70). The volume integrals over the j_z leads to a fixed value $q = \int j_z dV$ for the charges q and q^m and their values should be quantized according to Dirac's monopole argument.[24] Whatever the initial value of j^e_z or j^m_z, both should be conserved. The continuity condition $d \cdot j = \partial_0 j_0 + \nabla \vec{j} = 0$ is obtained by taking the 4-divergence:

$$d^2\beta_s d^2\alpha_s - d^2\alpha_s d^2\beta_s + d\beta_s \cdot d^2 d\alpha_s - d\alpha_s \cdot d^2 d\beta_s = \mathbf{u}_i \cdot d^2\mathbf{u}_r - \mathbf{u}_r \cdot d^2\mathbf{u}_i = 0 \tag{71}$$
$$d^2\beta_q d^2\alpha_q - d^2\alpha_q d^2\beta_q + d\beta_q \cdot d^2 d\alpha_q - d\alpha_q \cdot d^2 d\beta_q = \mathbf{v}_i \cdot d^2\mathbf{v}_r - \mathbf{v}_r \cdot d^2\mathbf{v}_i = 0 \tag{72}$$
$$d^2\beta_q d^2\alpha_s - d^2\alpha_s d^2\beta_q + d\beta_q \cdot d^2 d\alpha_s - d\alpha_s \cdot d^2 d\beta_q = \mathbf{v}_i \cdot d^2\mathbf{u}_r - \mathbf{u}_r \cdot d^2\mathbf{v}_i = 0 \tag{73}$$
$$d^2\beta_s d^2\alpha_q - d^2\alpha_q d^2\beta_s + d\beta_s \cdot d^2 d\alpha_q - d\alpha_q \cdot d^2 d\beta_s = \mathbf{u}_i \cdot d^2\mathbf{v}_r - \mathbf{v}_r \cdot d^2\mathbf{u}_i = 0 \tag{74}$$

where we have used the two 4-component complex vectors $\mathbf{u}$ and $\mathbf{v}$ again. It now appears that we can simply put in a mass term:

$$\mathbf{u}_i \cdot (d^2 + m^2)\mathbf{u}_r - \mathbf{u}_r \cdot (d^2 + m^2)\mathbf{u}_i = 0 \tag{75}$$
$$\mathbf{v}_i \cdot (d^2 + m^2)\mathbf{v}_r - \mathbf{v}_r \cdot (d^2 + m^2)\mathbf{v}_i = 0 \tag{76}$$
$$\mathbf{v}_i \cdot (d^2 + m^2)\mathbf{u}_r - \mathbf{u}_r \cdot (d^2 + m^2)\mathbf{v}_i = 0 \tag{77}$$
$$\mathbf{u}_i \cdot (d^2 + m^2)\mathbf{v}_r - \mathbf{v}_r \cdot (d^2 + m^2)\mathbf{u}_i = 0 \tag{78}$$

which is fulfilled if our wave function Ψ is a solution to the Klein-Gordon equation $(d^2 + m^2)\Psi = 0$ and as a consequence the current density j, written as a quantum mechanical probability density current, Eq. (54), is valid. So our condition also says that the Ψ we have proposed is a valid quantum mechanical wave function! Guidance by the Clifford algebra $C\ell_{1,3}$ and starting from a complex 4-potential $A = \alpha d\beta$, appears to properly fuse electromagnetism (the inhomogeneous Maxwell equations with two oppositely charged sources) and relativistic quantum mechanics (the Klein Gordon equation). Moreover, with $d^2 + m^2 = (\mathbf{i}d - m)(\mathbf{i}d + m)$ our Ψ may equally well satisfy the Dirac equation $(\mathbf{i}d - m)\Phi = 0$, or when $m = 0$, it satisfies the wave equation, of course. This is good news! So far our Ψ was based entirely on electromagnetism. Indeed, we have now shown that the same Ψ that follows from electromagnetism also produces a current that looks identical to a quantum probability current and simultaneously obeys the Klein-Gordon equation. By comparing the probability current density of bosons, Eq. (44), with Eq. (30), that of fermions

$$j_\mu = \mathbf{i}\,(\psi^* \partial_\mu \psi - (\partial_\mu \psi^*)\psi) \quad \text{bosons} \tag{79}$$
$$j_\mu = -\phi^* \gamma_0 \gamma_\mu \phi = \phi^* e_{\mu 0} \phi \quad \text{fermions} \tag{80}$$

we see that the current produced by our Ψ is clearly of bosonic nature and it will *certainly not* obey the Dirac equation. In stead, the two currents may be interpreted as stemming from a singlet spin-entangled pair of a fermion (e) and its anti-particle (p):

$$\Psi = \frac{1}{\sqrt{2}}[\Phi_{p\downarrow}\Phi_{e\uparrow} \pm \Phi_{e\downarrow}\Phi_{p\uparrow}] \tag{81}$$

with the Φ obeying the Dirac equation $(\mathbf{i}d - m)\Phi = 0$ or $(\mathbf{i}\gamma^\mu \partial_\mu - m)\Phi = 0$ instead. Indeed, the charge density ρ seems to be composed of two objects with complex conjugate wave functions, which would indicate charge conjugation and that we are dealing with a particle-anti-particle pair.[29] In the context of this paper we may think of it as a knot and anti-knot or as an example, to represent an electron (e) and positron (p) with spin up and down. So we see that the bosonic state vector Ψ should hold the sum or difference of two so-called products states of two fermionic state vectors Φ_e and Φ_p belonging to the two particles, where we identify the 4-component vector Φ_o with the particle and let the label $_\uparrow$ indicate its spin state.[30]

The 4-current density has the correct form of quantum mechanical probability density, but we have not derived a quantum of action $\hbar$. The current consists of two parts: $j = j_a + j_b$. It appears that j_a is a complex conjugate of j_b, and hence it may be interpreted as the 4-current of the anti-particle. What seems to be described here is the opposite of the reaction from Eq. (1), namely pair creation! It may be any kind of knot or particle pair (charged or neutral, boson or fermion). So far, the theory is completely general, regardless of what kind of particle, topological solution, or electromagnetic knot is flowing as carrier of j_a or j_b, and it seems to describe the stage at which the knotted solutions will play. To produce the knots, both nonlinearity and foliation are required.

8. NONLINEARITY FROM NULL CONDITIONS AND ENERGY CONSERVATION

In this section we are coming back to the the null conditions (which are Lorentz invariant conditions) $F^2 = 0$ which says that only the *total* field is nul. More explicitly, we have $F^2 = (F_1 + F_2)^2 = E^2 - B^2 + 2e_{0123}\vec{E}\cdot\vec{B}$ and want $F_1 \neq F_2$, not $F_1 = F_2$ as is used for the knotted light in the literature[8] where $F_1^2 = 0$ and $F_2^2 = 0$. Instead we have $F_1^2 + F_2^2 = 0$ with $F_1^2 \neq 0$ and $F_2^2 \neq 0$ and also $F_1 \neq F_2$, the latter being absolutely required (but not sufficent) for the possibility of charged (knotted) fields. We expect to find a nonlinear equation that may have solutions that reveal the general topology and structure of the charged knots, but more importantly it may set a scale for their masses and hopefully reveals a mechanism for a stabilization force . We find that:

$$\begin{aligned} \vec{E}_1\cdot\vec{E}_2 = \vec{B}_1\cdot\vec{B}_2 &= \partial_0\alpha_s\nabla\beta_s\cdot(\nabla\alpha_s\times\nabla\beta_q) - \partial_0\alpha_q\nabla\beta_q\cdot(\nabla\alpha_q\times\nabla\beta_s) \\ &+ \partial_0\beta_s\nabla\alpha_s\cdot(\nabla\alpha_q\times\nabla\beta_s) - \partial_0\beta_q\nabla\alpha_q\cdot(\nabla\alpha_s\times\nabla\beta_q) \end{aligned} \tag{82}$$

$$\begin{aligned} \vec{E}_1\cdot\vec{B}_2 = -\vec{E}_2\cdot\vec{B}_1 &= \partial_0\alpha_s\nabla\beta_s\cdot(\nabla\alpha_q\times\nabla\beta_q) + \partial_0\alpha_q\nabla\beta_q\cdot(\nabla\alpha_s\times\nabla\beta_s) \\ &- \partial_0\beta_s\nabla\alpha_s\cdot(\nabla\alpha_q\times\nabla\beta_q) - \partial_0\beta_q\nabla\alpha_q\cdot(\nabla\alpha_s\times\nabla\beta_s) \end{aligned} \tag{83}$$

which means that the cross terms vanish and hence that $F^2 = (F_1 + F_2)^2 = F_1^2 + F_2^2$, and we are left with:

$$\vec{E}_1^2 + \vec{E}_2^2 - \vec{B}_1^2 - \vec{B}_2^2 = 0 \tag{84}$$

$$\vec{E}_1\cdot\vec{B}_1 + \vec{E}_2\cdot\vec{B}_2 = 0 \tag{85}$$

Those nonlinear conditions are part of the set of equations that may describe the lump of energy forming the knot or particle. An equivalent form can be found in terms of α and β:

$$(d\alpha^*\cdot d\alpha)(d\beta\cdot d\beta^*) = (d\alpha\cdot d\beta^*)^2 \tag{86}$$

$$(d\alpha\cdot d\beta^*)^2 = (d\alpha^*\cdot d\beta)^2 \tag{87}$$

where the dot means taking the scalar product. Note that:

$$(d\alpha)^2 = (\partial_0\alpha_s)^2 - (\nabla\alpha_s)^2 + (\partial_0\alpha_q)^2 - (\nabla\alpha_q)^2 \tag{88}$$

$$d\alpha^*\cdot d\alpha = (\partial_0\alpha_s)^2 - (\nabla\alpha_s)^2 - (\partial_0\alpha_q)^2 + (\nabla\alpha_q)^2 + 2e_{0123}(\partial_0\alpha_s\partial_0\alpha_q - \nabla\alpha_s\cdot\nabla\alpha_q) \tag{89}$$

$$\begin{aligned} d\alpha^*\cdot d\beta &= (\partial_0\alpha_s\partial_0\beta_s - \nabla\alpha_s\cdot\nabla\beta_s + \partial_0\alpha_q\partial_0\beta_q - \nabla\alpha_q\cdot\nabla\beta_q) \\ &+ e_{0123}(\partial_0\alpha_s\partial_0\beta_q - \nabla\alpha_s\cdot\nabla\beta_q - \partial_0\alpha_q\partial_0\beta_s + \nabla\alpha_q\cdot\nabla\beta_s) \end{aligned} \tag{90}$$

We can put Eqs. (86) and (87) in a form of complex vectors and their Hermitian conjugates:

$$(\widetilde{\mathbf{a}}\cdot\mathbf{a})(\mathbf{b}\cdot\widetilde{\mathbf{b}}) = (\mathbf{a}\cdot\widetilde{\mathbf{b}})^2 = (\widetilde{\mathbf{a}}\cdot\mathbf{b})^2 \tag{91}$$

where $\mathbf{a} = d\alpha$ and $\mathbf{a}^\dagger = (d\alpha)^\dagger = e_0 d\alpha^* e_0 = e_0\widetilde{\mathbf{a}}e_0$ and where $\widetilde{\mathbf{a}} = d\alpha^* = e_0\mathbf{a}^\dagger e_0$.
Although each single one of these conditions is already complete, the question remains whether there are more

useful ones we can derive from them. In any case we have some nonlinear equation involving the quantum mechanical state vectors, it looks like a coupled quadratic equation of **a** and **b**. In the section "Discussion" we may come back to this.

Additional nonlinear conditions may be found from the conservation of energy; with the energy density $U = (\vec{E}^2 + \vec{B}^2)/2$ and Poynting vector $\vec{S} = \vec{E} \times \vec{B}$ following from $FF^\dagger$, its conservation is expressed by the continuity equation $d(FF^\dagger) = 0$, which is nonlinear in the fields F.

Then there is the possibility of a non-vanishing (complex) scalar field Υ, related to the gauge. A more general invariant than $F^2 = 0$ would have to be used. From the literature,[21] but more specifically from as yet unpublished work,[22] the proper Lorentz invariant would be:

$$I = \left(\Upsilon_s^2 - \Upsilon_q^2 - \vec{E}^2 + \vec{B}^2\right) + 2e_{0123}\left(\Upsilon_s \Upsilon_q - \vec{E} \cdot \vec{B}\right) \tag{92}$$

This unpublished work will be submitted for publication. It contains a completely general result for invariants in the space-time algebra $C\ell_{1,3}$ and a discussion on the consequences of it not being a division algebra.

Where the linear equations describe the playground of the knots, the nonlinear conditions should provide stability and length of scale, but not likely without some foliation of the complex fields α and β to obtain a non-trivial topological structure.

9. TOWARDS TOPOLOGICAL SOLUTIONS

We have now constructed a sufficiently good back bone to turn to the topology and look for specific knotted solutions. If we introduce a mapping $\alpha \to \alpha^p$ and $\beta \to \beta^q$ with $p \neq q$, when p and q are > 1, a foliation of the solutions occurs which should give us the topological fields and the particles we hope for! Earlier we have written: $\alpha = \alpha_s + e_{0123}\alpha_q$ and $\beta = \beta_s + e_{0123}\beta_q$. Now we can replace this with

$$\alpha^p = (\alpha_r + e_{0123}\alpha_i)^p = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \alpha_r^k (e_{0123}\alpha_q)^{n-k} = \alpha_s + e_{0123}\alpha_q \tag{93}$$

and leave all our previous equations unaltered. Kedia et all[8] give examples of how we might proceed from here. We will, however, leave the exciting and important endeavor of finding knotted solutions for later work.

10. DISCUSSION AND SUMMARY

We have taken the algebra $C\ell_{1,3}$ of space-time and derive Maxwell's electromagnetism from a 4-potential $A = \alpha d\beta$. The remarkable and very useful property of the space-time algebra is that it provides, simultaneously, a proper geometrical basis for electromagnetism and a Dirac algebra to do relativistic quantum mechanics. The utility of this is obvious if we want to describe, in its finest detail, the fundamental process of, for example, the annihilation of electrons and positrons to pure gamma radiation: $e^+ + e^- \longrightarrow \gamma + \gamma$. How is it that the (quantum) current is transmuted in pure radiation?

We have introduced a 4-potential $A = \alpha d\beta$ inspired by Bateman. Bateman starts from the gradients of two complex fields α and β to obtain a general Rieman-Silberstein vector $F = \nabla\alpha \times \nabla\beta$ which has been shown to allow (non-charged) topological solutions of the electromagnetic field F under a complex scalar map such as $\alpha \to \alpha^p$ and $\beta \to \beta^q$.[8] On top of these results, our potential $A = \alpha d\beta$ may produce charged solutions.

In approach it appears that the fields split naturally: $F = F_1 + F_2$, for which F_1 has $\nabla \cdot B_1 = 0$ and $\nabla \cdot E_1 = 0$. Other authors[8] have only considered $F = F_1 = F_2$, and hence $E_1 = E_2$ which implies that also $\nabla \cdot E_2 = 0$, hence NO CHARGE is possible for their knots.

The total field must form a proper spinor, an invariant, and hence we must have that $F^2 = 0$ such that F is a null-vector. This appears to reduce to $(F_1 + F_2)^2 = F_1^2 + F_2^2 = 0$ (and not $F_1^2 = F_2^2 = 0$). As a result we obtain a non-linear equation that may or may not provide stability of the solution.

It is now possible to express the inhomogeneous Maxwell equations as $dF = dF_1 + dF_2 = j = j_a + j_b$ such that $j_a = \Psi_a^* d\Psi_a - \Psi_a d\Psi_a^*$. This is a new result. Both F and Ψ can be expressed in terms of the two complex fields α and β. A current density j_a of this quantum mechanical form can only follow from some non-abelian mathematics and not from Maxwell's equations when written in the usual way and derived from a 4-vector potential only.

However, to obtain the quantum mechanical expression for j, some extra conditions need to be fulfilled. It is required that Ψ obeys a coupled equation $\Psi_b \cdot (d^2 + m_a^2)\Psi_a = \Psi_a \cdot (d^2 + m_b^2)\Psi_b$. This is true when either a Klein-Gordon or Dirac equation is obeyed: $(d^2 + m^2)\Psi = (id - m)(id + m)\Psi = 0$. We may interpret this result as follows: there are two currents, of opposite or neutral charge and spin. One possibility is that there are two scalar particles (spin-0 bosons), with or without charge. Another possibility is that we are looking at an entangled state of two spin-1/2 fermions, a particle and its anti-particle.

Although the current density has the correct form of quantum mechanical probability density, but there is no constant of action $\hbar$ or quantization of charge e. Such constants may however appear in a topological theory from the periodicity of the specific solutions.[11,26] So far, we have only described the playground of such solutions and the theory is completely general, regardless of what kind of particle, topological solution, or electromagnetic knot is flowing as carrier of j_a or j_b.

In this paper, a lot is still missing. We have not calculated any topological solutions and have not looked at solutions in an external field. We have hardly looked at gauge conditions and not proven the consistency of the back bone of the theory. If, however, in the end it all holds up, it will be a giant step in the direction of true unification of electrodynamics and relativistic quantum mechanics. Potentially, it may do even more: we have touched upon the weak interaction with the incorporation of the neutrino equation and the knotted flows of energy may effectively feel a strong interaction when pulling one part of a loop tightens another part. The role of linked field lines is to provide topological robustness against decay, representing an interaction that goes beyond electromagnetism as it is commonly known.

11. CONCLUSIONS

In this paper, we have set up the backbone for a unifying theory of quantum mechanics and electromagnetism. It is set up in such a way that it may support topological solutions of the electromagnetic fields. These are electromagnetic knots, but formulated on a basis of the space-time algebra $C\ell_{1,3}$, a real Dirac algebra, such that quantum aspects may be taken into account. It may be valid for all charged or even for uncharged elementary particles (neutral currents, neutrino's). Our work hinges on the recent successes obtained with knotted light and on a long history of topological electromagnetism and particle models.

We have shown that the method introduced by Bateman, where (topological) electromagnetic fields are generated from $F = \nabla\alpha \times \nabla\beta$ can be generalized to a 4-potential A with $dA = d(\alpha d\beta) = F = F_1 + F_2$ and that this allows massive and potentially charged solutions in case $F_1 \neq F_2$. Maxwell's equations appear to be inhomogeneous: $dF = j_a + j_b$ with the pair of 4-currents is a of quantum mechanical form $j = \Psi^* d\Psi - \Psi d\Psi^*$, but under the condition that the associated currents are scalar (spin-0) and described by some wave function $\Psi = d\alpha + \mathbf{i} d\beta$ that must fulfill the Klein-Gordon equation $(d^2 + m^2)\Psi = 0$. On top of this, the two currents may be interpreted as stemming from an entangled pair of a fermionic knot ant its anti-knot, $\Phi = \Psi_{p\downarrow}\Psi_{e\uparrow} \pm \Psi_{e\downarrow}\Psi_{p\uparrow}$ hence obeying the Dirac equation $(\mathbf{i}d - m)\Phi = 0$. This connects relativistic quantum mechanics and electromagnetism in the most intimate way thinkable. Note however that this will not affect the validity of quantum electrodynamics as a theory of interaction of charges by means of photons, but it may put in place a better foundation for it. What remains is to find solutions (that will now hold in both theories) for the topological structure of the sources j in terms of fields. A nonlinear equation $F_1^2 + F_2^2 = 0$ is given by the condition that F and Ψ must be null to make them proper Lorentz invariant spinors. What needs to be proven and is crucial, is that the topology and nonlinearity will work together in such a way that an internal force balance may be found. Perhaps, then we find that the fundamental "stuff" everything is made of[14] is, in mathematical terms, just the combination of complex fields $d\alpha$ and $d\beta$ that produces the potential A, the electromagnetic fields F, currents J and wave functions Ψ. The associated charged knots (the elementary particles) will have a finite size with as a direct consequence that divergences will disappear from the theory.

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