Inversion and general invariance in Space-Time

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Abstract

A general formula for division in a relativistic Clifford-Dirac algebra is derived. Where division is undefined turns out, in many cases, to correspond to areas of physical interest, such as the light cone, invariant quantities in electromagnetism, and the basis set of quantities in the Dirac equation. Apart from such areas, where there has already been significant development in science, new sets of inter-related quantities, involving the spin and the total energy for example, are sugested as possibilities for further investigation and development.

Keywords: non-division algebra, invariants, inverses, scaling, Dirac algebra, Clifford algebra, geometric algebra, special relativity, generalized covariance, rest mass, light cone, diamond conjugate

1. Introduction

The Clifford-Dirac algebra provides a basis for spin-1/2 relativistic quantum mechanics, Maxwell electromagnetism and advanced geometry. Despite the fact that the algebra has proven to be a powerful basis of physical theory it, in common with any Dirac algebra, is not a division algebra. In this paper we will study division in the Clifford-Dirac algebra $C\ell_{1,3}$ and derive the most general case for inversion. This then leads to the notion of the general invariants that, through their implied symmetries, necessarily lie at the basis of all physical theory.

Relativistic algebras, such as any Dirac algebra and the Clifford algebra $C\ell_{1,3}$ in particular, are not division algebras in that there are areas other than zero where division is not defined. Though this may seem an undesirable feature mathematically, and indeed is known to cause problems in certain areas[10], physically it is required to properly parallel relativistic space-time. In particular, for non-zero 4-vectors on the light cone where the invariant interval goes to zero, division by this quantity is manifestly undefined. The Clifford algebra $C\ell_{1,3}$, often denoted the space-time algebra or just the STA for short, has been designed to parallel as closely as possible the nature of space-time [1, 2, 3, 4, 5, 6, 7, 8]. More recently other authors have noted that, because $C\ell_{1,3}$ is isomorphic to the base elements of a particular Dirac algebra [9], another appropriate name for it is a Clifford-Dirac algebra and this name is also in current usage [10].

In any event, even though the algebra is not a division algebra, it appears to be of utility, not only in describing spin-1/2 [5, 11] but also, for example, in describing aspects of physics such as the Maxwell equations[3, 13]. In this context, the development leads, not only to a description of the physics which is comparable to that of other methods, but that also is in some respects more elegant. In particular, the formulation leads to all four Maxwell equations at once [2, 4, 8, 12, 13], rather than to the pair of inhomogenous equations for the field and the homogeneous equations for the dual field separately as is the case in the more usual textbook approach [15]. How can this be? How is it that a non-division algebra can successfully describe wide areas of physics? The physical reason is that the world observed in experiment *does* scale relativistically. That scaling, for the mass of a particle as it approaches light-speed for example, tends

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to infinity, and hence its inverse quantity tends to zero. The quantities describing dynamics in Maxwell and Dirac theory, however, are 4-vector differentials, for which the scaling of each component taken separately is precisely unity. It is not in the case of individual elements, but in combinations of elements where division becomes undefined. In fact one may turn the perspective around, and say that, for properly relativistic algebras *only* quantities with this unit property may be important for a local description of dynamics - as they lead to possible unitary operators which conserve important quantities such as energy and momentum.

Now one comes to the physical utility of inverses (and hence division) in this context. Division may seem familiar, and is so for simple numbers: the inverse of three is a third. What, physically, is the inverse of space? time? space-time? space divided by time? space-time on the light-cone? If one can find an inverse, the product of this with its starting quantity leads to a unit Lorentz scalar. It may be suspected, as indeed turns out to be the case, that finding such combinations may lead, in turn, to unitary processes which leads in turn to"allowed" and interesting physics. The extension of the unit relativistic vectors of space and time leads to a rich set of combinations of derived elements, corresponding to combinations of physical areas, volumes and point, as well as base lines, where division is undefined.

The structure of this paper is as follows. For those unfamiliar with Dirac-Clifford algebras the essential properties are described[1, 2, 3, 4, 5, 6, 7, 8, 13]. Inverses are found for various quantities of importance, including the general case for the algebra considered here. It is shown that the areas where division is undefined correspond to null-hyperplanes which cut through the extended structure of the algebra. It is argued that many of these null-hyperplanes correspond to limiting cases of physical interest, such as the zero-length interval (null-vector) of space-time in Einstein's special relativity, the corresponding case in energy-momentum and invariant quantities important in electromagnetism. Some of these particular cases are discussed. One of these corresponds to most of the terms in the Dirac equation, but suggests an extra missing term which may be that leading to the proper description of charge as well as half-integral spin. This may help complete a program to describe the nature of charge which Dirac pursued unsuccessfully in the fifties[20]. Other combinations with important invariants, hitherto having had little or no attention in the literature, are uncovered which may be of interest for future study. The form of the general case is discussed, and it is argued that this may lead to energy minima with non-zero rest masses and hence an understanding of the underlying nature of the potential at the root of the Higgs mechanism.

2. The Clifford-Dirac algebra

Dirac developed his algebra in the first instance to pass to a linearisation of the energy momentum Hamiltonian in relativistic quantum mechanics. Clifford algebras have been used, through their geometric product over the basis vectors of space and time, to represent the full range of boosts and non-commuting rotations between them. The sub-algebra of the Dirac γ-matrix algebra excluding γ_5 is isomorphic to the Clifford algebra $C\ell_{1,3}$. It is this real Clifford-Dirac algebra that we will investigate here. Note that the standard Dirac γ-matrices, not including γ_5, are a representation of this Clifford algebra, but any specific matrix representation is irrelevant to any of the arguments which follow. A contravariant 4-vector A is written

$$A = (A_0, \boldsymbol{A}) = \gamma_\mu A_\mu = \gamma_0 A_0 + \gamma_1 A_1 + \gamma_2 A_2 + \gamma_3 A_3 \tag{1}$$

with the A_μ being real coefficients. Note that lower indices are used in the case of contravariant vectors, as this simplifies the notation for squared quantities. Summation is implied for repeated Greek indices which run from 0 to 3. Latin indices run from 1 to 3.

The 16 terms of the full geometric product between two 4-vectors is defined as

$$AB = A \circ B + A \wedge B \tag{2}$$

the first part of which is the symmetric part and corresponds to the 4-vector scalar product in this simple case:

$$A \circ B = \frac{1}{2}(AB + BA) \tag{3}$$

It is worth noting that in the present paper the symmetric part of the geometric product $A \circ B$ is denoted by a small circle in order to avoid any confusion with the dot product: $\boldsymbol{x} \cdot \boldsymbol{y}$, the scalar or inner product between ordinary 3-vectors (denoted by boldface).

The second part of the product, the antisymmetric part, behaves in some respects (at least between two vectors) like the usual Heaviside-Gibbs cross product of 3-space, $\times$, but is denoted here by the wedge symbol [4]:

$$A \wedge B = \frac{1}{2}(AB - BA) \tag{4}$$

For definiteness, the generators of the algebra are mapped onto the unit basis vectors of Minkowski space-time as

$$\gamma_0 = c\hat{t}, \quad \gamma_1 = \hat{x}, \quad \gamma_2 = \hat{y}, \quad \gamma_3 = \hat{z} \tag{5}$$

This choice will only be of importance for the physical interpretation of our results. The anti-commutator of the basis vectors is

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_{\mu\nu} + \gamma_{\nu\mu} = 2g_{\mu\nu}\mathbb{1} \ , \tag{6}$$

where the metric tensor $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(+ - - -)$ has the Lorentz metric,

$$\gamma_0^2 = -\gamma_i^2 = 1 \quad , \qquad i = \{1 \dots 3\} \tag{7}$$

Note that a convention $\gamma_\mu \gamma_\nu = \gamma_{\mu\nu}$ is adopted, not only in an effort to keep the terms compact, but also to make explicit that these are new elements in a group of sixteen orthogonal elements [6].

The square of a vector A gives precisely the Lorentz-invariant scalar product:

$$\begin{aligned} A^2 &= \gamma_\mu A_\mu \gamma_\nu A_\nu = \gamma_0^2 A_0^2 + \gamma_1^2 A_1^2 + \gamma_2^2 A_2^2 + \gamma_3^2 A_3^2 \\ &= A_0^2 - A_1^2 - A_2^2 - A_3^2 \end{aligned} \tag{8}$$

and the proper invariant interval ds of space-time is

$$(ds)^2 = (dx_0)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 \tag{9}$$

where ds is both positive-definite and time-like (γ_0) for subluminal world lines.

Starting with the unit basis elements γ_μ, using the antisymmetric product, Eq. (4), unit elements of higher grade can be formed. There are 6 independent terms of the form $\gamma_\mu \gamma_\nu$ which we abbreviate with $\gamma_{\mu\nu}$, the bivector unit basis elements. Just as the γ_i form a basis for translations in Minkowski 4-space, the higher grade elements γ_{i0} form the basis elements of boosts (Lorentz transformations) and the γ_{jk} the basis elements of rotations, with their proper non-commutative properties included. Note that $\gamma_{\mu\nu} = -\gamma_{\nu\mu}$ for $\mu \neq \nu$; any exchange of adjacent indices generates a factor of minus one. There are four independent trivectors (the pseudo 4-vector basis elements) of the form $\gamma_\lambda \gamma_\mu \gamma_\nu = \gamma_{\lambda\mu\nu}$, and a single independent quadrivector γ_{0123}, the pseudoscalar. Together with the generator basis vectors γ_μ and the scalar $\gamma_0^2 = 1$ we have 16 linearly independent unit elements which, together with their counterparts with negative sign, form an algebraic group of 32 elements. The real algebra with this group requires only the positive 16 unit basis because the minus sign is absorbed in the real coefficients. So called multivectors can be formed using these elements. The most general multivector $\Psi = s + v + b + r + t + q$, containing all basis elements, is defined as

$$\Psi = \mathbb{1}s_0 + \gamma_0 v_0 + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \boldsymbol{v} + \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{pmatrix} \boldsymbol{b} + \begin{pmatrix} \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix} \boldsymbol{r} + \begin{pmatrix} \gamma_{023} \\ \gamma_{031} \\ \gamma_{012} \end{pmatrix} \boldsymbol{t} + \gamma_{123} t_0 + \gamma_{0123} q_0 \tag{10}$$

Here we have distinguished terms which naturally have three components, such as the Lorentz "boosts" $\boldsymbol{b}$ and "rotations" $\boldsymbol{r}$ as well as the 3-vector parts of 4-vectors with boldface characters. For example the 3-vector part of the 4-vector v is denoted $\boldsymbol{v}$. Each of these corresponds to the column vector preceding it and the notation implies an inner product between the two. See, for example Eq. (21) in what follows. The advantage of the 3-component column vector notation is that it makes explicit the "3-space plus 1-time" structure of (the 4-dimensional generators of) the algebra, including their reflection properties (time-reversal and parity). Also it allows for a seamless transformation to the familiar Heaviside-Gibbs vector algebra notation, as will become apparent in the next section. By keeping the unit basis elements explicit, we not only allow for distinction of the grade of a multivector component, but also find these distinctions to be of value in the classification of inverses.

A short calculation shows that $\gamma_0^2 = \gamma_{i0}^2 = \gamma_{123}^2 = 1$, $\gamma_i^2 = \gamma_{ij}^2 = \gamma_{0ij}^2 = \gamma_{0123}^2 = -1$. Of the 10 elements which square to -1, not one commutes with all other elements, that is, none behave like the complex number $\mathbf{i} = \sqrt{-1}$. There is no γ_5 unless one explicitly adds the unit imaginary. That is, the Dirac γ-matrices are representations of the group that forms the basis for the Clifford algebra of spacetime $C\ell_{1,3}$ [1, 5, 6, 9], but the Dirac matrix algebra $M_4(\mathbb{C})$ (the algebra of complex 4×4 matrices) is the complexification of both the spacetime algebra: $\mathbb{C} \otimes C\ell_{1,3} \simeq M_4(\mathbb{C})$ and the Majorana algebra $\mathbb{C} \otimes C\ell_{3,1} \simeq M_4(\mathbb{C})$ [6]. For the even subalgebra $\{1, \gamma_{i0}, \gamma_{jk}, \gamma_{0123}\}$, the quadrivector γ_{0123} takes the role of the unit imaginary number $\sqrt{-1}$ because it commutes with all the even elements. The sixteen element set generated from the basis γ_μ on the Lorentz metric $(+\ -\ -\ -)$ forms a "geometric" Dirac algebra, the Clifford algebra of space-time $C\ell_{1,3}$.

For the quotients $\gamma^\mu = 1/\gamma_\mu$, which correspond to the covariant basis vectors, we have

$$\gamma^0 = \gamma_0, \qquad \gamma^i = -\gamma_i \tag{11}$$

As a consequence of the quotient, the vector differential operator is a reciprocal vector and so has opposite space sign to the vector:

$$\begin{aligned} d = \frac{\partial}{\gamma_\mu \partial x_\mu} &= \gamma^\mu \partial_\mu = g^{\mu\nu}\gamma_\mu \partial_\nu \\ &= \gamma_0\partial_0 - \gamma_1\partial_1 - \gamma_2\partial_2 - \gamma_3\partial_3 = \gamma_0\partial_0 - \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \boldsymbol{\nabla} \end{aligned} \tag{12}$$

Clearly, the operation of this differential operator d on some multivector Ψ also results in a change of grade. The scalar 4-space Laplacian operator (d'Alembertian) is:

$$d^2 = d \circ d = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 = \partial_0^2 - \boldsymbol{\nabla}^2 \tag{13}$$

it does not change the grade of any multivector.

3. Brief introduction to the notation

In this paper we make use of the real Dirac-Clifford algebra $C\ell_{1,3}$, also known the space-time algebra (STA) or geometric algebra[4], to express electromagnetic fields and current densities as well as 4-component Dirac wave functions. Central to this algebra is the so-called geometric product of two vectors A and B, which is defined as $AB = A \circ B + A \wedge B$ which consists of a symmetric part $A \circ B = (AB + BA)/2$ and an anti-symmetric part $A \wedge B = (AB - BA)/2$. First we introduce the unit basis vectors of space and time e_μ, which behave exactly like the Dirac matrices γ_μ with their commutation relations $e_\mu e_\nu - e_\nu e_\mu = 2g_{\mu\nu}$ with $g_{\mu\nu} = \mathrm{diag}(+\ -\ -\ -)$. By multiplication, a total of 16 so-called (unit) multivectors can be formed, for example $e_{0123} \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ which has the property that $e_{0123}^2 = -1$ and it takes the role of the unit imaginary, despite the fact that it only commutes with the even sub group $\{\mathbb{1}, e_{\mu\nu}, e_{0123}\}$. The conventional Dirac algebra is the complexification of the real geometric algebra $C\ell_{1,3}$. We will only use real numbers and give up complex numbers and it is the price we have to pay in order to keep a geometric interpretation and proper basis for Maxwell's equations on the one hand and a relativistic quantum mechanics on the other hand. The four-potential $A = (A_0, \boldsymbol{A})$ is

$$A = e_0A_0 + e_1A_1 + e_2A_2 + e_3A_3 \equiv e_0A_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \boldsymbol{A} \tag{14}$$

where we have adopted a mixed notation using 3-space vectors to separate the time-like (e_0) and space-like (e_i) parts of the four vector (e_μ). This allows us to use the standard vector calculus notation and the Dirac algebra simultaneously. To many readers, this will appear to be helpful in recognizing known physics even if geometric algebra is new to them. The column notation shows the unit basis vectors (e_i, e_j, e_k), with $i = \{i, i0, jk, 0jk\}$ (cyclic), as they project onto the unit vectors (e_x, e_y, e_z) of 3-space. The most general multivector, having all possible components, is $M = s + v + b + r + t + q$ or

$$M = s_0 + e_0 v_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \boldsymbol{v} + \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} \boldsymbol{b} + \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \boldsymbol{r} + \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} \boldsymbol{t} + e_{123} t_0 + e_{0123} q_0 \tag{15}$$

The letters refer to the nature the of the basis element: scalar, vector (polar vector), bivector (boost (polar vector) and rotor (axial vector)), trivector (pseudo vector or axial vector) and quadrivector (pseudoscalar). In our notation, it is this implicit projection that makes the connection between the Dirac matrices and the geometry of space-time. The differential operator is

$$d = e_0 \partial_0 - e_1 \partial_1 - e_2 \partial_2 - e_3 \partial_3 \tag{16}$$

The differential operating on the 4-potential is given by the geometric product $dA = d \cdot A + d \wedge A$ and this can be written explicitly as

$$dA = \partial_0 A_0 + \nabla \cdot \boldsymbol{A} - \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\partial_0 \boldsymbol{A} + \nabla A_0) - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \nabla \times \boldsymbol{A} \tag{17}$$

which consists of a scalar part $L = d \cdot A$ and a bivector part $F = d \wedge A$, we write $dA = L + F$. In case we put $L = 0$ we have the Lorenz gauge condition:

$$L = \partial_0 A_0 + \nabla \cdot \boldsymbol{A} = 0 \tag{18}$$

The equivalent of the Faraday or field-strength tensor $F^{\mu\nu}$ is represented by the double bivector F:

$$F = \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} \boldsymbol{E} - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} e_{10} \\ e_{20} \\ e_{30} \end{pmatrix} (\boldsymbol{E} + e_{0123} \boldsymbol{B}) \tag{19}$$

The bivector F is also referred to as the Riemann-Silberstein vector. For a general multivector M, the hermitian conjugate is $M^\dagger = e_0 \widetilde{M} e_0$, where the tilde means reversal, for example $\widetilde{AB} = BA$ in case of two vectors.
The derivative of F yields the field differentials of Maxwell's equations, $dF = j$:

$$\begin{aligned} dF &= e_0 \nabla \cdot \boldsymbol{E} + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (\nabla \times \boldsymbol{B} - \partial_0 \boldsymbol{E}) + e_{123} \nabla \cdot \boldsymbol{B} - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (\nabla \times \boldsymbol{E} + \partial_0 \boldsymbol{B}) \\ &= e_0 j_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \boldsymbol{j} = j \end{aligned} \tag{20}$$

where j is the 4-current density, the source term which must be put in by hand. It is also the part that is associated with the self-energy problem of electrical (point) charge. It is the bold purpose of this paper to find a detailed mathematical description of the structure of charged particles (and their motion) as knotted solutions of fields, potentials, wave functions and energy flow. We will show that we can write both sides of Maxwell's equations in terms of two complex scalar fields α and β: $dF(\alpha, \beta) = j(\alpha, \beta)$. In doing so, j may take the form of a quantum probability 4-current with wave function $\Psi(\alpha, \beta)$ that obeys relativistic quantum mechanics.

4. The Maxwell equations

Starting with a vector 4-potential $A(x)$ defined over all space-time x and using the algebra with the vector differential introduced above, Maxwell's equations may be derived in a particularly beautiful and compact way. This result has been derived by other authors [3, 4, 6, 8]. Our purpose here is twofold; firstly to define physical quantities for later use in the discussion and secondly to give an example of our explicit but compact 3-vector column notation introduced in the previous section. In what follows natural units are used, ε_0, $\hbar$ and c are set equal to unity.

Let the 4-potential be $A = (A_0(t, \boldsymbol{x}), \boldsymbol{A}(t, \boldsymbol{x}))$ with A_0 the scalar potential and $\boldsymbol{A}$ the vector potential. In accordance with the previous section:

$$A = \gamma_\mu A_\mu = \gamma_0 A_0 + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \boldsymbol{A} \tag{21}$$

The 4-derivative is $dA = d \circ A + d \wedge A$. To aid visualisation, this may be written in terms of the familiar 3-space forms, such as $\boldsymbol{A}$, the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, and the standard dot and cross product, whilst the full 4-space algebra is maintained by means of the positional column notation introduced above for the proper components. With these conventions, the 16 $(= 1 + 3 + 3 \cdot 2 + 3 \cdot 2)$ terms of the full product dA may be written as

$$dA = \partial_0 A_0 + \boldsymbol{\nabla} \cdot \boldsymbol{A} - \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{pmatrix} (\partial_0 \boldsymbol{A} + \boldsymbol{\nabla} A_0) - \begin{pmatrix} \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix} \boldsymbol{\nabla} \times \boldsymbol{A} \tag{22}$$

which is the sum of a scalar part L and a bivector part F, so we can write $dA = L + F$, with

$$L = d \circ A = \partial_0 A_0 + \boldsymbol{\nabla} \cdot \boldsymbol{A} \tag{23}$$

The scalar L is intimately related to the gauge. For an arbitrary scalar function Λ the gauge freedom is expressed by the transformation $A \to A + d\Lambda$ and hence $L \to L + d^2\Lambda$. Setting $L = 0$ (for all coordinates) corresponds to the Lorenz gauge condition.

In Eq. (22) we can identify, in the usual way, the electric field $\boldsymbol{E} = -\partial_0 \boldsymbol{A} - \boldsymbol{\nabla} A_0$ and the magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. Together these terms form a six-component object known as the Riemann-Siberstein vector which we denote by F and which corresponds to the antisymmetric Faraday or field-strength tensor $F^{\mu\nu}$ [15], but here it takes the spinor form [16, 17]:

$$F = \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{pmatrix} \boldsymbol{E} - \begin{pmatrix} \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{pmatrix} (\boldsymbol{E} + \gamma_{0123} \boldsymbol{B}) \tag{24}$$

In Eq. (24) the electric and magnetic fields have a bivector form, a boost γ_{i0} and a rotor γ_{ij} respectively, rather than appearing as a set of tensor components.

Consider the dynamics of dA, and set $L = 0$ at all coordinates, so that $dL = 0$. Usually at this point, a 4-vector current density source term j is introduced:

$$d(dA) = d^2 A = d(L + F) = dL + dF = j \tag{25}$$

so that

$$dF = j \tag{26}$$

which represents all the Maxwell equations [8]. In the full geometrical product for Eq. (26), $dF = d \circ F + d \wedge F$, the vector part and the trivector part, are identified as the homogeneous and inhomogeneous Maxwell equations respectively:

$$d \circ F = j \tag{27}$$

$$d \wedge F = 0 \tag{28}$$

The trivector part Eq. (28) has no source term, expressing the absence of magnetically charged monopoles, consistent with a vector-potential description where $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. As in Eq. (22), Eqs. (27) and (28) may be expanded in terms of the full 4-space products, and terms may be gathered in the familiar 3-space quantities to give

$$\gamma_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \gamma_0 j_0 \tag{29}$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} (\boldsymbol{\nabla} \times \boldsymbol{B} - \partial_0 \boldsymbol{E}) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \boldsymbol{j} \tag{30}$$

$$\gamma_{123} \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{31}$$

$$-\begin{pmatrix} \gamma_{023} \\ \gamma_{031} \\ \gamma_{012} \end{pmatrix} (\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_0 \boldsymbol{B}) = 0 \tag{32}$$

which we recognise immediately as the full set of Maxwell equations, with all the correct signs, though with the proper multivector form of the equations within the algebra made explicit. Starting from the 4-potential A and retaining *all* products within the algebra, both the homogeneous Eqs. (28), (31) and (32) and inhomogeneous Eqs. (27), (29) and (30) equations are contained in a single equation, Eq. (26), without the need to introduce a separate dual field. This is in contrast to other developments using either standard notation [15] or the algebra of forms [18], where in both approaches two equations are required to cover the homogeneous and inhomogeneous Maxwell equations respectively. By carrying every part of the product the full Maxwell equations have been obtained, with all the correct signs, and nothing more.

5. On invariants and the hyperplanes where division is not defined

Let us now pass to the main purpose of this paper, a discussion of where and how division is, and is not, defined within the Clifford-Dirac algebra at hand. The aim is to distinguish those special multivectors Ψ where a "multiplicative division" or inverse does not exist so that an inverse Ψ^{-1} cannot be found such that $\Psi\Psi^{-1} = 1$, and hence where division is not defined [10].

In many algebras, including the real, the complex and the quaternion algebras, zero is the only element which has no inverse. Here there are many more combinations for which an inverse does not exist. These are referred to as null-hyperplanes, since they correspond to objects of zero length, a so-called null-vector (such as a Riemann-Silberstein vector for the electromagnetic field), as also proposed by Kramers [16] and Weyl [**?**]. We first discuss some specific familiar cases and then go on to present a general form for the inverse.

First consider the 4-vector case:

$$\Psi = v = \gamma_0 v_0 + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \boldsymbol{v} \tag{33}$$

$$\Psi^{-1} = v/v^2 = v/(v_0^2 - \boldsymbol{v}^2) = \frac{\Psi}{v_0^2 - \boldsymbol{v}^2} = \Psi/\tau^2 \tag{34}$$

Note, for the case of the space-time coordinates $v_0 = ct$ and $\boldsymbol{v} = \boldsymbol{x}$, the divisor corresponds to the invariant interval squared τ^2 and that all inverses are scaled, correctly, according to this interval. On the lightcone this interval goes to zero, and hence there is no inverse if $v_0^2 - \boldsymbol{v}^2 = 0$. That is the plane where division is undefined corresponds exactly to the physical limitations imposed by the speed of light. This theme will be picked up again in the discussion in the context of that special form of division, the differential. There are, of course, many interesting invariants with the vector form. For example the corresponding invariant in the case of the 4-vector potential is a charge invariant [20].

Consider further the combination of a scalar and a Lorentz boost:

$$\Psi = s + b \tag{35}$$

$$\Psi^{-1} = (s - b)/(s_0^2 - \boldsymbol{b}^2) \tag{36}$$

This is the form for the energy and momentum density in the field, in which case the divisor corresponds to an invariant mass m_0. This will be expanded on in the discussion. This has no inverse if $s_0^2 - \boldsymbol{b}^2 = 0$ and corresponds to the lightcone as well. The divisor is a true scalar in the algebra and, as such, is invariant under a Lorentz transformation, a property shared with the pseudoscalar, which will appear in some of the more general cases which follow. Note that the inverse vector is another vector in the same direction whereas in the case of scalar plus boost the inverse aquires a minus sign in the spatial component.

It is possible to extend the vector null-hyperplane to include the scalar and the pseudoscalar as well:

$$\Psi = s + v + q \tag{37}$$

$$\Psi^{-1} = (s - v - q)/(s_0^2 - v_0^2 + \boldsymbol{v}^2 + q_0^2) \tag{38}$$

This has no inverse if $v_0^2 - \boldsymbol{v}^2 = s_0^2 + q_0^2$. In the context of electromagnetism it contains the gauge term (scalar) as well as the quadrivector (the dual gauge). We see that the addition of a gauge field shifts the null-multivectors off the lightcone. This has applications in the description of massive, rather than massless systems.

The combination with all the elements that square to +1 also has a null-hyperplane:

$$\Psi = s + \gamma_0 v_0 + b + \gamma_{123} t_0 \tag{39}$$

$$\Psi^{-1} = (s - \gamma_0 v_0 - b - \gamma_{123} t_0)/(s_0^2 - v_0^2 - \boldsymbol{b}^2 - t_0^2) \tag{40}$$

There is no inverse, for example, if $s_0^2 - \boldsymbol{b}^2 = v_0^2 + t_0^2$. Multivectors with all elements squaring to +1 will prove essential in the derivation of a completely general inverse as will be shown by the end of this section. Also, The same set of basis elements are those necessary in the splitting of the relativistic Klein-Gordon equation to obtain the linear form of the Dirac equation as will be returned to in the discussion.

Consider the following:

$$\Psi = s_0 + \gamma_0 v_0 + \gamma_{123} t_0 + \gamma_{0123} q_0 \tag{41}$$

$$\Psi^{-1} = \frac{s_0 - \gamma_0 v_0 - \gamma_{123} t_0 - \gamma_{0123} q_0}{s_0^2 - v_0^2 - t_0^2 + q_0^2} \tag{42}$$

This has no inverse if $s_0^2 + q_0^2 = v_0^2 + t_0^2$, and connects all the single element "time like" parts of the algebra. Dynamics over this set would imply an interaction between time and the gauge fields, which, it may be speculated, could lead to extra quantisation conditions on any full set of interacting fields [21].

In view of the previous examples, it is now clear that the following formula helps in finding Ψ^{-1} in many (simple) cases:

$$\Psi^{-1} \simeq \Psi^{\diamond}/(s_0^2 - v_0^2 + \boldsymbol{v}^2 - \boldsymbol{b}^2 + \boldsymbol{r}^2 + \boldsymbol{t}^2 - t_0^2 + q_0^2) \tag{43}$$

Here we have defined the "diamond" conjugate of a multivector Φ as

$$\Phi^{\diamond} = 2\langle\Phi\rangle_s - \Phi \tag{44}$$

where $\langle\Phi\rangle_s$ is the scalar part of Φ. For the special case of the single-grade multivectors the validity of Eq. (43) is trivial. Note that

$$\begin{aligned}
\Psi\Psi^{\diamond} &= s_0^2 - v_0^2 + \boldsymbol{v}^2 - \boldsymbol{b}^2 + \boldsymbol{r}^2 + \boldsymbol{t}^2 - t_0^2 + q_0^2 + 2\gamma_0 \boldsymbol{r}\cdot\boldsymbol{t} \\
&+ 2\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}(t_0\boldsymbol{r} - \boldsymbol{b}\times\boldsymbol{t}) - 2\begin{pmatrix}\gamma_{10}\\ \gamma_{20}\\ \gamma_{30}\end{pmatrix}(q_0\boldsymbol{r} + \boldsymbol{v}\times\boldsymbol{t}) \\
&- 2\begin{pmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{pmatrix}(v_0\boldsymbol{t} - t_0\boldsymbol{v} - q_0\boldsymbol{b}) - 2\begin{pmatrix}\gamma_{023}\\ \gamma_{031}\\ \gamma_{012}\end{pmatrix}(v_0\boldsymbol{r} + \boldsymbol{v}\times\boldsymbol{b}) \\
&- 2\gamma_{123}\boldsymbol{v}\cdot\boldsymbol{r} + 2\gamma_{0123}\boldsymbol{b}\cdot\boldsymbol{r}
\end{aligned} \tag{45}$$

The method of finding the inverse by using Eq. (43) is guaranteed only if $\Psi\Psi^\diamond$ is a real number (a scalar) so that $\Psi^{-1} = \Psi^\diamond/\Psi\Psi^\diamond$. It can, however, also be used iteratively on $\Psi\Psi^\diamond$ etc. To make clear the connection with the physics we switch notation to that used for the field quantities in the section on the Maxwell equations, $L = s$, $F = b + r = \boldsymbol{E} - \boldsymbol{B}$, see also Eqs. (23) and (24). Doing this for the complete even subgroup leads to:

$$\Psi = s + b + r + q = L + F + q \tag{46}$$

$$\begin{aligned} \Psi^{-1} &= \frac{(L+F^\dagger - q)(L - F^\dagger - q)(L - F + q)}{(L^2 + \boldsymbol{E}^2 + \boldsymbol{B}^2 + q_0^2)^2 - 4[(L\boldsymbol{E} + q_0\boldsymbol{B})^2 + (\boldsymbol{E}\times\boldsymbol{B})^2]} \\ &= \frac{(L - F + q)[L^2 - \boldsymbol{E}^2 + \boldsymbol{B}^2 - q_0^2 + 2\gamma_{0123}(q_0 L - \boldsymbol{E}\cdot\boldsymbol{B})]}{(L^2 - \boldsymbol{E}^2 + \boldsymbol{B}^2 - q_0^2)^2 + 4(q_0 L - \boldsymbol{E}\cdot\boldsymbol{B})^2} \end{aligned} \tag{47}$$

The invariant divisor in Eq. (47) brings out an important invariant in electromagnetism [11], which will be returned to in the discussion.

If Ψ is a multivector, $\Psi^\dagger$ corresponds to its Hermitian conjugate $\Psi^\dagger = \gamma_0\widetilde{\Psi}\gamma_0$, where $\widetilde{\Psi}$ is the reversed ordering of all multivector components of Ψ. The † operation reverses the sign of all basis elements of the algebra which square to -1, so that in the product $\Psi\Psi^\dagger$ all "oscillating" terms are quenched.

$$\begin{aligned} \Psi\Psi^\dagger &= s_0^2 + v_0^2 + \boldsymbol{v}^2 + \boldsymbol{b}^2 + \boldsymbol{r}^2 + \boldsymbol{t}^2 + t_0^2 + q_0^2 \\ &+ 2\gamma_0(s_0v_0 + \boldsymbol{r}\cdot\boldsymbol{t} + t_0q_0 - \boldsymbol{v}\cdot\boldsymbol{b}) \\ &+ 2\begin{pmatrix}\gamma_{10}\\ \gamma_{20}\\ \gamma_{30}\end{pmatrix}(s_0\boldsymbol{b} - q_0\boldsymbol{r} - \boldsymbol{v}\times\boldsymbol{t} + v_0\boldsymbol{v} + t_0\boldsymbol{t} - \boldsymbol{b}\times\boldsymbol{r}) \\ &+ 2\gamma_{123}(s_0t_0 - \boldsymbol{v}\cdot\boldsymbol{r} - v_0q_0 - \boldsymbol{b}\cdot\boldsymbol{t}) \end{aligned} \tag{48}$$

Note that $\Psi\Psi^\dagger$ contains no more than just the six multivector components that square to $+1$, and this appears to be a good starting point for further reduction to a scalar (real number). Using $\Psi\Psi^\dagger$ and Eq. (39) and Eq. (40) the general case of the inverse of Ψ now follows as

$$\Psi^{-1} = \frac{\Psi^\dagger(2\langle\Psi\Psi^\dagger\rangle_s - \Psi\Psi^\dagger)}{\Psi\Psi^\dagger(2\langle\Psi\Psi^\dagger\rangle_s - \Psi\Psi^\dagger)} = \frac{\Psi^\dagger(\Psi\Psi^\dagger)^\diamond}{\Psi\Psi^\dagger(\Psi\Psi^\dagger)^\diamond} = \frac{\Psi^\dagger\Phi^\diamond}{\Phi\Phi^\diamond} \tag{49}$$

where the denominator is always a true (Lorentz) scalar (a real number). This is the first important new result of this paper. We have defined $\Phi \equiv \Psi\Psi^\dagger$ and used Eq. (44). It also follows that

$$(\Psi\Psi^\dagger)^{-1} = \Phi^{-1} = \frac{\Phi^\diamond}{\Phi\Phi^\diamond} \tag{50}$$

Note that Ψ^{-1} and Φ^{-1} have the same null-hyperplanes. Note also that $\Phi^\dagger = (\Psi\Psi^\dagger)^\dagger = \Psi\Psi^\dagger = \Phi$ and the product $\Phi\Phi^\diamond = \Phi^\diamond\Phi$ is an invariant scalar. This scalar can be expressed in terms of the components of Ψ:

$$\begin{aligned} \Phi\Phi^\diamond &= (s_0^2 + v_0^2 + \boldsymbol{v}^2 + \boldsymbol{b}^2 + \boldsymbol{r}^2 + \boldsymbol{t}^2 + t_0^2 + q_0^2)^2 \\ &- 4(s_0v_0 + \boldsymbol{r}\cdot\boldsymbol{t} + t_0q_0 - \boldsymbol{v}\cdot\boldsymbol{b})^2 \\ &- 4(s_0\boldsymbol{b} - q_0\boldsymbol{r} - \boldsymbol{v}\times\boldsymbol{t} + v_0\boldsymbol{v} + t_0\boldsymbol{t} - \boldsymbol{b}\times\boldsymbol{r})^2 \\ &- 4(s_0t_0 - \boldsymbol{v}\cdot\boldsymbol{r} - v_0q_0 - \boldsymbol{b}\cdot\boldsymbol{t})^2 \end{aligned} \tag{51}$$

Hence

$$\Phi\Phi^\diamond \equiv \langle\Psi\Psi^\dagger\rangle_s^2 - 4N_\diamond^2 = (\langle\Psi\Psi^\dagger\rangle_s + 2N_\diamond)(\langle\Psi\Psi^\dagger\rangle_s - 2N_\diamond) \tag{52}$$

where the positive scalar $N_\diamond^2$ is defined as

$$\begin{aligned} N_\diamond^2 &= (s_0v_0 + \boldsymbol{r}\cdot\boldsymbol{t})^2 + (t_0q_0 - \boldsymbol{v}\cdot\boldsymbol{b})^2 + (s_0t_0 - \boldsymbol{v}\cdot\boldsymbol{r})^2 + (v_0q_0 + \boldsymbol{b}\cdot\boldsymbol{t})^2 \\ &+ (s_0\boldsymbol{b} - q_0\boldsymbol{r} - \boldsymbol{v}\times\boldsymbol{t})^2 + (v_0\boldsymbol{v} + t_0\boldsymbol{t} - \boldsymbol{b}\times\boldsymbol{r})^2 \end{aligned} \tag{53}$$

The second important new result of this paper is that, for the general case, all null-hyperplanes are given by $\langle \Psi\Psi^\dagger \rangle_s = \pm 2N_\diamond$ and that $\Phi\Phi^\diamond$ is the difference of two positive definite scalars which represents a general invariant in this formulation.

The invariant quantity $\Phi\Phi^\diamond$ is a Lorentz scalar and we can define it to be the fourth power of some effective invariant (or rest) mass μ_0 (one factor μ_0 for every Ψ). That rest mass is not static, but arises from the internal dynamics of Φ. Note that the effective rest mass of light speed objects is zero and is real for sub-luminal objects (and imaginary for super-luminal objects). One defines:

$$\mu_0^4 \equiv \Phi\Phi^\diamond = \langle \Psi\Psi^\dagger \rangle_s^2 - 4N_\diamond^2 = \mu^4 - 4N_\diamond^2 \tag{54}$$

where $\langle \Psi\Psi^\dagger \rangle_s$, since it is a scalar, may be interpreted to be the square of the total mass μ or, equivalently, the total energy. This may be verified for the simple example $\Psi = s+b$ where $\Phi\Phi^\diamond = (s_0^2 - b^2)^2 = (E^2 - p^2)^2 = E_0^4$, and where $\mu^4 = \langle \Psi\Psi^\dagger \rangle_s^2 = (s_0^2 + b^2)^2$. We may also write:

$$\mu_0^4 \equiv \langle \Psi\Psi^\dagger \rangle_s^2 - 4N_\diamond^2 = 2\langle \Psi\Psi^\dagger \rangle_s \Psi\Psi^\dagger - (\Psi\Psi^\dagger)^2 \tag{55}$$

where in a simplified case $2\Psi\Psi^\dagger$ may be interpreted to be the square of some field ϕ:

$$\mu_0^4 = 2\langle \Psi\Psi^\dagger \rangle_s \Psi\Psi^\dagger - (\Psi\Psi^\dagger)^2 = \mu^2\phi^2 - \lambda\phi^4 = V(\phi), \quad \text{with} \quad \lambda = \frac{1}{4} \tag{56}$$

This means that the rest mass of a particle follows from a scalar potential $\mu_0^4 = V(\phi)$ which depends on the field ϕ and a mass μ. A potential of quartic form may take the form of a Mexican hat and hence to so-called spontaneous symmetry breaking, such that ground-state level of the energy, and hence its associated mass, are non-zero. Interestingly the field $\phi^2/4 = \Psi\Psi^\dagger$ does not need to be scalar, for example if $\Psi = \Psi^\dagger$, then the condition is already fulfilled. Hence this means that any general Ψ that only contains elements that square to plus one is good, such as for the Dirac operator! We may want to compare $V(\phi)$ to the Higgs potential and μ to the Higgs mass, but no extra Higgs field ϕ needs to be postulated.

6. Discussion

Any relativistic algebra spans a space which is laced by a network of overlapping hyperplanes where division is not defined. There is no such analogy in real, complex or quaternionic algebras where division is defined everywhere except for zero itself. It is remarkable that many of these combinations correspond to cases of primary physical importance.

In the previous section it has been argued that the way in which the inverse vector scales as the lightcone is approached is just the way space and time scale in special relativity, with division being undefined on the lightcone itself. In the light of this there is an argument to be made for the reverse proposition: that any algebra which does not scale relativistically should not be considered sufficiently general to be considered, by itself, a good description of the underlying nature of space and time. Much of the current thinking in quantum mechanics is in terms of complex algebra, but complex numbers themselves do not have this scaling property and the state space of quantum mechanics, the Hilbert space, is non-relativistic.

It has been shown that the non-definition of division everywhere is no impediment to the development of a powerful vector differential algebra. Indeed, the subtlety and beauty of the interactions between the non-commuting basis elements and the 4-vector derivative leads to all the Maxwell equations in a single step and with a single field (no dual field) in the example earlier. We now try to shed some light on how and why the vector differential Eq. (12) should prove so potent in the description of that subset of reality described by the Maxwell equations.

Consider an arbitrary space-time 4-vector x :

$$x = \gamma_0 x_0 + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \boldsymbol{x} \tag{57}$$

with inverse

$$x^{-1} = x/x^2 = x/(x_0^2 - \boldsymbol{x}^2) \tag{58}$$

This has no inverse if the divisor $x_0^2 - \boldsymbol{x}^2 = 0$. Note again that the scalar divisor in Eq. (58) is just the square of the invariant interval and corresponds exactly to the change in scale of length and time (rulers and clocks) in special relativity.

As an illustration of the utility of this we investigate the limit of a divisor as we approach the null-hyperplane formed by the lightcone. For the 4-vector x of Eq. (57) define $(\Delta s_\tau)^2 = (\Delta ct)^2 - (\Delta x_1)^2 - (\Delta x_2)^2 - (\Delta x_3)^2$ where Δs_τ is the length scale of the 4-dimensional interval. We follow Feynman Vol. II, Chap. 26, Ref. [23]. Defining $d_0 = \gamma_0 \partial_0$, passing from Δs_τ to its limit $ds = c d\tau$, defining $d_\tau = d/d\tau$, and eliminating position variables in favour of velocity variables $dx_i = v_i dx_0$ leads to:

$$\begin{aligned} c\,d\tau &= \gamma_0\, dx_0 \sqrt{1 - (\partial_0 x_1)^2 - (\partial_0 x_2)^2 - (\partial_0 x_3)^2} \\ &= \gamma_0\, dx_0 \sqrt{1 - (v/c)^2} = \gamma_0 c \gamma^{-1}\, dt \end{aligned} \tag{59}$$

Hence we find the invariant scalar operator

$$d_\tau = \gamma_0 c \gamma d_0 \tag{60}$$

This reveals within the Clifford-Dirac algebra the *combined* role of the unit time vector γ_0 and the γ-factor ($\gamma = 1/\sqrt{1-(v/c)^2}$) to relate the proper time τ to the time t in some specific frame. The differential operator for time d_0 and the covariant derivative d_τ differ by a grade γ_0 (d_τ is a scalar, d_0 is a time-like vector), and this clarifies the distinction between 3-dimensional space, with t a mere parameter, and 4-dimensional space-time.

The extraordinary utility of this derivative, and its scaling properties, is best illustrated with a simple example. Consider the derivative with respect to time in some specific frame of the 4-vector x defined as

$$c\, d_0 x = c - \gamma_{10} v_1 - \gamma_{20} v_2 - \gamma_{30} v_3 \tag{61}$$

This is not in vector form, but in a combination of scalar and bivector in which the 3-velocity part is a bivector, and the effect of the metric has been to reverse its sign. The proper 4-velocity is given by

$$d_\tau x = \gamma(\gamma_0 c + \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3) \tag{62}$$

which is a true 4-vector.

The beauty and simplicity of the invariant derivative in maintaining the form of the multivector acted upon may lead to a view that it should be adopted as the preferred form for describing dynamics, but nature acts otherwise. The disturbing feature of Eq. (61), that the vector derivative altered the multi-vector plane of the result, was ubiquitous in the derivation of the Maxwell equations above. How and why does this all work? The answer lies partly in the subtle way in which the vector derivative transforms, and transforms again the multi-vectors amongst each other, and partly in the fact that in each step the derived fields have many transformation properties in common with their precursors. In particular there are areas where division is not defined for the scalar-bivector combination introduced in Eq. (61), which are related to another important invariant, the invariant mass (the rest mass), as is now discussed.

Consider the field product $\Psi\Psi^\dagger$ for $\Psi = F$

$$\frac{1}{2} F F^\dagger = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{pmatrix} (\boldsymbol{E} \times \boldsymbol{B}) \tag{63}$$

The scalar part represents the energy density of the electromagnetic field and the bivector part the Poynting vector, which represents electromagnetic momentum density. As has been discussed in the previous section, this combination has a null-hyperplane which behaves similarly in many respects to that of the vector. To see this consider the case of Eq. (35) for $\Psi = s + b$ which has divisor $s_0^2 - \boldsymbol{b}^2$. The divisor here corresponds to the invariant mass density, and appears undefined in the case of a zero mass density. Since (rest) massless particles and fields are lightspeed this again corresponds to the lightcone. The scalar plus bivector combination is not a 4-vector, but its divisor scales in the same way as that of a 4-vector under Lorentz transformations. Taking a time derivative of this form yields a true 4-vector. It can also be transformed into a true 4-vector by multiplying by a unit vector in the time direction, though this is a frame dependent operation. This operation has the effect of mapping the vectors to the scalar plus bivector combination and vice-versa.

Using the general formula, it is possible to find the following simple cases which include the rotor:

$$\Psi = \gamma_0 v_0 + r + \gamma_{123} t_0 + q \tag{64}$$

$$\Psi^{-1} = \frac{(\gamma_0 v_0 + r + \gamma_{123} t_0 - q)}{(v_0^2 + t_0^2 + \boldsymbol{r}^2 - q_0^2)} \tag{65}$$

and the simple case

$$\Psi = s + r \tag{66}$$

$$\Psi^{-1} = (s - r)/(s_0^2 + \boldsymbol{r}^2) \tag{67}$$

This would have no inverse if $s_0^2 + \boldsymbol{r}^2 = 0$, which would imply $s_0^2 = 0$ and $\boldsymbol{r}^2 = 0$, so Ψ would be zero anyway. This means that there is no null-hyperplane in this case, and hence division is defined for all combinations of such elements except zero itself. This special combination, which forms a sub-group within the algebra, is isomorphic to the quaternions which themselves form a division ring. Physically this means that processes of this nature are unrestricted, unitary and have no limit. One may go round and round as much as one wishes, without having to scale or to transform through the scalar.

Also note the following cases

$$\Psi = b + q \tag{68}$$

$$\Psi^{-1} = (b - q)/(\boldsymbol{b}^2 + q_0^2) \tag{69}$$

and

$$\Psi = \gamma_0 v_0 + \begin{pmatrix} \gamma_{023} \\ \gamma_{031} \\ \gamma_{012} \end{pmatrix} \boldsymbol{t} \tag{70}$$

$$\Psi^{-1} = (\gamma_0 v_0 - \begin{pmatrix} \gamma_{023} \\ \gamma_{031} \\ \gamma_{012} \end{pmatrix} \boldsymbol{t})/(v_0^2 + \boldsymbol{t}^2) \tag{71}$$

for which division is always defined.

As a further example of the physical utility of these null-hyperplanes within the Clifford-Dirac algebra, it is instructive to consider the "null vectors" of Kramers [16]. For these we have $F^2 = 0$, c.f. Eq. (47):

$$FF = F^2 = \boldsymbol{E}^2 - \boldsymbol{B}^2 + 2\gamma_{0123} \boldsymbol{E} \cdot \boldsymbol{B} \tag{72}$$

This requires $\boldsymbol{E}^2 = \boldsymbol{B}^2$ and $\boldsymbol{E} \perp \boldsymbol{B}$, corresponding to the free electromagnetic wave and it corresponds to the case where there is no inverse for

$$\Psi^{-1} = FF^\dagger F^\dagger/((\boldsymbol{E}^2 - \boldsymbol{B}^2)^2 + 4(\boldsymbol{E} \cdot \boldsymbol{B})^2) \tag{73}$$

Again, the null-vector of Eq. (72) appears as a divisor. In each of the cases above it would seem that the physics is constrained by the existence of each of these null-hyperplanes, and conversely, that the investigation of the corresponding invariant divisors may throw further light on the physics.

Another null-hyperplane of potential physical importance is that with respect to the scalar, the tri-vector and the pseudoscalar:

$$\Psi = s + t + q \tag{74}$$
$$\Psi^{-1} = (s - t - q)/(s_0^2 - t_0^2 + \boldsymbol{t}^2 + q_0^2) \tag{75}$$

which is precisely analogous to the case of the vector, scalar and the pseudoscalar of Eq. (37). The tri-vector quantities here represent a product of a momentum density, with a perpendicular vector. This is analogous to an angular momentum density[13].

In the preceding, we have started with a Dirac algebra and have looked for inverses whose product yielded a simple scalar. In a sense, this is the reverse process to that followed by Dirac. He started with a square root scalar operator and was forced to introduce a Dirac algebra to linearise it. Introducing the scalar operator $\mathcal{H}$, the classical relativistic Hamiltonian and demanding it be linear in the components of the momentum p_1, p_2 and p_3 we obtain:

$$\mathcal{H}/c = \sqrt{m^2c^2 + \boldsymbol{p}^2} = \gamma_0 mc + \gamma_{10}p_1 + \gamma_{20}p_2 + \gamma_{30}p_3 \tag{76}$$

Together with the energy p_0, this led to his relativistic quantum mechanical operator equation

$$(p_0 + \gamma_0 mc + \gamma_{10}p_1 + \gamma_{20}p_2 + \gamma_{30}p_3)\,|\Psi\rangle = 0 \tag{77}$$

Note that the resulting operator only contains basis elements that square to +1. Hence, by demanding this equation to be roughly equivalent to the classical scalar equation, Dirac obtained his non-commutative algebra. Originally, the notation $\alpha_i = \gamma_{i0}$ and $\alpha_m = \gamma_0$ was used. Squaring the original relativistic equation that contains the classical Hamiltonian appears to be equivalent to the multiplication of the linearized equation Eq. (77) with the conjugate operator $p_0 - \gamma_0 mc - \gamma_{10}p_1 - \gamma_{20}p_2 - \gamma_{30}p_3$.

In the context of the previous section this operator is recognized as the "diamond" conjugate of the linear operator in Eq. (77). Those multivectors with elements squaring to plus one, let's call those Φ, which have the property $\Phi^\dagger = \Phi$, appear to play a central role within the algebra: The same pair of conjugate multivectors Φ and $\Phi^\diamond$ are essential both in forming the Dirac linear operator as well as in properly defining division and finding inverses within the space-time algebra.

In the context of the Dirac equation, the Dirac algebra is successful in describing, amongst many other things, half integral spin and the existence of the positron. It has been developed to be consistent with special relativity: invariant for scalars (s), covariant for vectors and tri-vectors (v, t), and with the proper transformations of the fields (r, b), and, of course, it is all of these things. Any relativistic algebra must necessarily contain a proper description, at the very least, of connections on the light cone with invariant interval zero. Comparing Eq. (77) with Eq. (39) and Eq. (40), one observes that all the terms squaring to positive unity are represented, except one, that corresponding to the directed volume element γ_{123}. The Dirac equation properly describes the half-integral spin, but says nothing about the charge, though Dirac tried to remedy this in later work[20]. The present authors have also made progress in trying to make this link[21, 14]. The relationship of γ_{123} to the angular momentum density is the same as that of the charge to the current density. In his paper on "A new classical theory of electrons", Dirac concluded[20]. "To make this passage one will presumably have to replace the square root in the Hamiltonian (18) with something involving spin variables ... This may be a difficult problem, but one can hope that its solution will lead to the quantization of electric charge and will fix e in terms of h." The term γ_{123} is a prime candidate for this transition.

7. Conclusions

A link has been suggested between the mathematical process of division and the physical processes of dynamics and transformation.

We have confirmed the work of other authors in that $C\ell_{1,3}$, used as a vector differential algebra, is a natural basis for the formulation of electromagnetism and leads to all the Maxwell equations, with the correct signs and without the need for a dual field. Further, the algebra is apt for describing the relativistic (Dirac) quantum mechanics of particles of half-integral spin.

By looking at the inverses of multivectors, where and how division is and is not defined within the Clifford-Dirac algebra has been investigated. It has been shown that the inverses scale relativistically in a way which parallels that which is observed in nature. Further, those sets of quantities that admit zero divisors embody some of the important invariant quantities of relativity and electromagnetism. For example we have found that division is not defined on the lightcone, and that the scalar divisor corresponds to the invariant interval. Taking the differential limit of the 4-vector as this divisor approaches the lightcone leads to the invariant proper derivative of special relativity. This derivative is, in turn, crucial to the dynamical description of light in the Maxwell equation, and of matter in the Dirac equation.

Further, other combinations parallel those familiar invariants of energy and momentum and the important invariants of the electromagnetic field. Another set parallels, with one exception, the set of quantities in the Dirac equation on which the 4-vector derivative, described above, acts. That missing quantity in the set, exposed by this analysis, may prove the essential root-Hamilitonian that Dirac sought to describe the underlying nature of charge in the further development of his famous theory.

In addition to all of this there is another simple hyperplane where division becomes undefined, involving the scalar (energy) and the angular momentum(spin) which has not yet been looked at at all. This is a subject which clearly merits further work.

The fact that many of these invariants may take the value zero for non-zero components, means that the algebra is laced with a network of null-hyperplanes where division is not defined. A new multivector conjugate and an explicit general formula for the inverse has been presented. In this case the divisor has a fourth-power character. The underlying elements are all dynamical, but each may be simplified by viewing it as possessing an "energy". Expressed in energetic terms, then, the total energy then has a "Mexican hat" form. This is similar to the form postulated for the Higgs mechanism, though here no extra potential is required as the terms prove to arise in the present treatment from existing elements of the even (Dirac) set of quantities.

We conclude that the fact the Clifford-Dirac algebra is not a division algebra, does not disqualify it as a candidate algebra of reality. On the contrary there is a case to be made for the reverse proposition: that the manner and areas where division is undefined in the algebra are precisely those required to properly parallel physical reality.

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8. Bibliography styles

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@article{mar00,
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