APPLICATIONS OF GRASSMANN'S EXTENSIVE ALGEBRA.

By Professor Clifford, *University College, London.*

I propose to communicate in a brief form some applications of Grassmann's theory which it seems unlikely that I shall find time to set forth at proper length, though I have waited long for it. Until recently I was unacquainted with the Ausdehnungslehre, and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers.* I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence upon the future of mathematical science.

The present communication endeavors to determine the place of Quaternions and of what I have elsewhere* called Biquaternions in the more extended system, thereby *explaining* the laws of those algebras in terms of simpler laws. It contains, next, a generalization of them, applicable to any number of dimensions; and a demonstration that the algebra thus obtained is always a compound of quaternion algebras which do not interfere with one another.

On the Relation of Grassmann's Method to Quaternions and Biquaternions; and on the Generalization of these Systems.

Following a suggestion of Professor Sylvester, I call that kind of multiplication in which the sign of the product is reversed by an interchange of two adjacent factors, *polar* multiplication; because the product ab has opposite properties at its two ends, so that $ab = -ba$. The ordinary or commutative multiplication I shall call *scalar*, being that which holds good of scalar numbers. These words answer to Grassmann's *outer* and *inner* multiplication; which names, however, do not describe the multiplication itself, but rather those geometrical circumstances to which it applies.

* Proceedings of the London Mathematical Society.

Consider now a system of n units $\iota_1, \iota_2, \ldots \iota_n$, such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$. For geometrical applications we may take these to represent points lying in a flat space of $n-1$ dimensions. A binary product $\iota_r \iota_s$ is then a unit length measured on the line joining the points ι_r, ι_s; a ternary product $\iota_r \iota_s \iota_t$ is a unit area measured on the plane through the three points, and so on. A linear combination of these units, $\Sigma a_r \iota_r$, $= \alpha$ suppose, represents a point in the given flat space of $n-1$ dimensions, according to the principles of the barycentric calculus, as extended in the Ausdehnungslehre of 1844.

In space of three dimensions we may take the four points $\iota_0, \iota_1, \iota_2, \iota_3$ so that $\iota_1, \iota_2, \iota_3$ are at an infinite distance from ι_0 in three directions at right angles to one another.

Now there are two sides to the notion of a product. When we say $2 \times 3 = 6$, we may regard the product 6 as a number derived from the numbers 2 and 3 by a process in which they play similar parts; or we may regard it as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers; in the latter view 3 is a number, but 2 is an operation, and the two factors play very distinct parts. *The Ausdehnungslehre is founded on the first view; the theory of quaternions on the second.* When a line is regarded as the product of two points, or a parallelogram as the product of its sides, the two factors are things of the same kind and play similar parts. But in such a quaternion equation as $q\rho = \sigma$, where ρ and σ are vectors, the quaternion q is an operation of turning and stretching which converts ρ into σ; it is a thing totally different in kind from the vector ρ. The only way in which the factors q and ρ can be taken to be of the same kind, is to regard ρ as itself a special case of a quaternion, viz: a rectangular versor. But in that case the expression does not receive its full meaning until we suppose a *subject* on which the operations ρ and q can be performed in succession.

The quaternion symbols i, j, k represent, then, *rectangular versors;* that is to say, they are operations which will turn a figure through a right angle in the three coordinate planes respectively. It follows that if either of them is applied twice over to the same figure, it will turn it through two right angles, or *reverse* it; we must therefore have $i^2 = j^2 = k^2 = -1$.

To compare these with the symbols for the four points $\iota_0, \iota_1, \iota_2, \iota_3$, let us suppose that i turns the line $\iota_0 \iota_2$ into $\iota_0 \iota_3$; that j turns $\iota_0 \iota_3$ into $\iota_0 \iota_1$; and that k

turns $\iota_0 \iota_1$ into $\iota_0 \iota_2$. The turning of $\iota_0 \iota_2$ into $\iota_0 \iota_3$ is equivalent to a translation along the line at infinity $\iota_2 \iota_3$. We may, therefore, write $i = \iota_2 \iota_3$, and so $j = \iota_3 \iota_1$, $k = \iota_1 \iota_2$. Now i turns $\iota_0 \iota_2$ into $\iota_0 \iota_3$; that is

$$i \cdot \iota_0 \iota_2 = \iota_0 \iota_3$$

or

$$\begin{aligned} \iota_0 \iota_3 &= \iota_2 \iota_3 \cdot \iota_0 \iota_2 \\ &= - \iota_2^2 \cdot \iota_0 \iota_3 . \end{aligned}$$

We are therefore obliged to write $\iota_2^2 = -1$, and in a similar way we may find $\iota_1^2 = \iota_3^2 = -1$.

This at once enables us to find the rules of multiplication of the i, j, k. Namely, we have

$$\begin{aligned} jk &= \iota_3 \iota_1 \cdot \iota_1 \iota_2 = \iota_2 \iota_3 = i \\ ki &= \iota_1 \iota_2 \cdot \iota_2 \iota_3 = \iota_3 \iota_1 = j \\ ij &= \iota_2 \iota_3 \cdot \iota_3 \iota_1 = \iota_1 \iota_2 = k \end{aligned}$$

and finally

$$ijk = \iota_2 \iota_3 \cdot \iota_3 \iota_1 \cdot \iota_1 \iota_2 = -1 .$$

In order, therefore, to bring the quaternion algebra within that of the Ausdehnungslehre, we have to make the square of each of our units equal to -1, as pointed out by Grassmann.* But I venture to differ from his authority in thinking that the quaternion symbols do not in the first place answer to the "Elementargrösse" of the Ausdehnungslehre, but to binary products of them; from which supposition, as we have seen, the laws of their multiplication follow at once.

It is quite true that in process of time the conception of a product as derived from factors of the same kind, and so of the product of two vectors as a thing which might be thought of without regarding them as rectangular versors, grew upon Hamilton's mind, and led to the gradual replacement of the units i, j, k by the more general selective symbols S and V. To explain the laws of multiplication of i, j, k on this view, we must have recourse to the theory of "Ergänzung," or, which comes to the same thing, *represent* an area ij by a vector k perpendicular to it. But the explanation in this case is by no means so easy; and it is instructive to observe that the distinction between a quantity and its "Ergänzung," *i. e.* between an area and its representative vector, which, for some purposes it is so convenient to ignore, has to be reïntroduced in physics. Thus Maxwell specially distinguishes the two kinds of vectors which he calls *force* and *flow*, and which in fact are respectively linear functions of the units and of their binary products.

* Math. Annalen.

We have regarded the symbols i, j, k as rectangular versors operating on the quantities $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. These quantities are unit lengths measured anywhere on the axes in the positive directions. They have magnitude, direction, and position, and are thus what I have called *rotors* (short for *rotators*) to distinguish them from *vectors*, which have magnitude and direction but no position. A vector is of the nature of the translation-velocity of a rigid body, or of a couple; it may be represented by a straight line of given length and direction drawn *anywhere*. A rotor is of the nature of the rotation-velocity of a rigid body, or of a force; it belongs to a definite axis. A vector may be represented as the difference of two points of equal weight (the vector ab may be written $b-a$); this is shewn by the principles of the barycentric calculus to represent a point of no weight at infinity. Accordingly the symbols $\iota_1, \iota_2, \iota_3$ may be taken to mean unit vectors along the axes. In fact, if we write $\iota_0+\iota_r=\alpha_r$, the points α will be situated on the axes at unit distance from the origin, and thus $\iota_r=\alpha_r-\iota_0$ will represent the unit vector from the origin to α_r.

The versors i, j, k will operate on these vectors in the same way as on the rotors $\iota_0\iota_1$, $\iota_0\iota_2$, $\iota_0\iota_3$. We find that $i\iota_2=\iota_2\iota_3 . \iota_2=\iota_3$, $j\iota_3=\iota_1$, $k\iota_1=\iota_2$. These rules of multiplication coincide with those for i, j, k if we write the latter in place of $\iota_1, \iota_2, \iota_3$. Thus we may use the same symbols to represent unit vectors along the axes and rectangular versors about them. But it is not in any sense true that the vectors ι_1, ι_2, ι_3 are identical with the areas $\iota_2\iota_3$, $\iota_3\iota_1$, $\iota_1\iota_2$; it is only sometimes convenient to forget the difference between ι_1 and $\iota_2\iota_3$.

In the elliptic or hyperbolic geometry* of three dimensions, the four points $\iota_0, \iota_1, \iota_2, \iota_3$ must be taken as the vertices of a tetrahedron self-conjugate in regard to the absolute, so that the distance between every two of them is a *quadrant*. The product of four points $\alpha\beta\gamma\delta$ will then consist of three kinds of terms; (1) terms of the fourth order, being $\iota_0\iota_1\iota_2\iota_3$ multiplied by the determinant of the coordinates of the four points, which is proportional to $\sin(\alpha,\beta)$ $\sin(\gamma,\delta)\cos(\alpha\beta,\gamma\delta)$; (2) terms of the second order, resulting from products of the form $\iota_0^2\iota_1\iota_2, =-\iota_1\iota_2$; (3) terms of order zero, resulting from products of the form ι_0^4, $\iota_0^2\iota_1^2$. Altogether we may arrange $\alpha\beta\gamma\delta$ in eight terms as follows:

$$\alpha\beta\gamma\delta=a+\Sigma b_{rs}\iota_r\iota_s+c\,\iota_0\iota_1\iota_2\iota_3. \quad [r,\ s \text{ different.}]$$

And it is now easy to see that the product of any *even* number of linear factors will be of the same form. This form is what I have called a *biquaternion*,

* Dr Klein's names for the geometry of a space of uniform positive or negative curvature. See Proc. Lond. Math. Soc.

and may be easily exhibited as such. Namely, let us write ω for $\iota_0\iota_1\iota_2\iota_3$; then we have

$$i = \iota_2\iota_3 \qquad j = \iota_3\iota_1 \qquad k = \iota_1\iota_2$$
$$\omega i = i\omega = \iota_1\iota_0\,,\ \omega j = j\omega = \iota_2\iota_0\,,\ \omega k = k\omega = \iota_3\iota_0$$
$$\omega^2 = 1\,.$$

Therefore, the product of any even number of factors greater than two is a linear function of 1, i, j, k, ω, ωi, ωj, ωk; that is to say, it is of the form $q + \omega r$, where q, r are quaternions. While the multiplication of ω with i, j, k is scalar, its multiplication with ι_0, ι_1, ι_2, ι_3 is polar. The effect of multiplying by ω is to change any system into its polar system in regard to the absolute.

The chief classification of geometric algebras is into those of *odd* and *even* dimensions. The geometry of an elliptic space of n dimensions is the same as the geometry of the points at an infinite distance in a flat or parabolic space of $n+1$ dimensions; the theory of *points* and *rotors* in the former is the same as that of vectors and their products in the latter. Each requires a geometric algebra of $n+1$ units. Thus the algebra of four units, leading as above to biquaternions, is either that of points and rotors in an elliptic space of three dimensions, or of vectors and their products in a flat space of four dimensions. All geometric algebras having an even number of units are closely analogous to it; of these I would point out particularly that of two units, belonging to the elliptic geometry of one dimension or to the theory of vectors in a plane. Let the units be ι_2, ι_3; then a product of any even number of linear functions must be of the form $a + b\iota_2\iota_3$. Let $i = \iota_2\iota_3$, then $i^2 = -1$; and such an even product is the ordinary complex number $a + bi$. In the method of Gauss every vector in the plane is represented by means of its ratio to the unit vector ι_2, that is to say, ι_2 and ι_3 are replaced by 1 and i. This gives an artificial but highly useful value for the product of two vectors. We might apply a similar interpretation to the algebra of four units, denoting the points $\iota_0, \iota_1, \iota_2, \iota_3$ by the symbols ω, i, j, k, and consequently their polar planes $\omega\iota_0$, $\omega\iota_1$, $\omega\iota_2$, $\omega\iota_3$ by the symbols 1, ωi, ωj, ωk; but I am not aware that any useful results would follow from this imitation of Gauss's plane of numbers.

Rules of Multiplication in an Algebra of n units.

In general, if we consider an algebra of n units, $\iota_1, \iota_2, \ldots \iota_n$, such that $\iota_r^2 = -1$, $\iota_r\iota_s = -\iota_s\iota_r$, a product of m linear factors will contain terms which are all of even order if m is even, and all of odd order if m is odd; for the substitution of -1 for any square factor of a term reduces the order of the term by 2.

A product of m units, all different, multiplied by any scalar is called a *term* of the order m. The sum of several terms of order m, each multiplied by a scalar, is a *form* of order m. The sum of several forms of different orders is a *quantity* and an even quantity when the forms are all of even order, an odd quantity when they are all of odd order. Thus the multiplication of linear functions of the units leads only to even quantities and odd quantities.

The square of a term of the m^{th} order is $+1$ or -1 according as the integer part of $\frac{1}{2}(m+1)$ is even or odd. For the product $\iota_1\iota_2 \ldots \iota_m\iota_1\iota_2 \ldots \iota_m$ is transformed into $\iota_1^2\iota_2^2 \ldots \iota_m^2$ by $\frac{1}{2}m(m-1)$ changes of consecutive factors, and therefore equals ± 1 according as $\frac{1}{2}m(m+1)$ is even or odd; which is equivalent to the rule stated.

The multiplication of a term P of order m by a term Q of order n, having k factors common, is scalar or polar according as $mn-k^2$ is even or odd. Let $P = CP'$ and $Q = CQ'$, where C, P', Q' have no common factor; then the steps from $CP'CQ'$ to $CP'Q'C$, $CQ'P'C$, $CQ'CP'$ require respectively $k(n-k)$, $(m-k)(n-k)$, $k(m-k)$ changes of consecutive factors; and the sum of these quantities is even or odd as $mn-k^2$ is.

The following cases are worth noticing:

(1) When two terms have no factor common, their multiplication is scalar except when they are both of odd order. (Case $k=0$).

(2) The multiplication of two even terms is scalar or polar according as the number of common factors is even or odd.

(3) If one of two terms is a factor in the other, the multiplication is scalar except when the first is odd and the second even.

Theory of Algebras with an odd number of units.

When the number of units is $n = 2m+1$, there are n terms of the order $n-1$, and all terms of even order can be expressed by means of these. For the product of any two of these terms is of the second order, since they must have $n-2$ factors common. We obtain in this way all the terms of the second order; and from them we can build up the terms of the fourth, sixth orders, etc. Let the product of all the units $\iota_1 \iota_2 \dots \iota_n$ be called ω, then these terms of the order $n-1$ shall be defined by the equations $k_r = \omega\iota_r$. It will follow that $k_1 k_2 \dots k_n = \mp 1$ according as m is even or odd, or, which is the same thing, according as the squares of the k are $+1$ or -1. By means of this formula, terms of order higher than m in the k, may be replaced by terms of order not higher than m. The multiplication of the k is always polar.

The terms of even order, regarded as compound units, constitute an algebra which is *linear* in the sense of Professor Peirce, viz: it is such that the product of any two of these terms is again a term of the system. The number of them is $2^{n-1} = 2^{2m}$; for the whole number of terms, odd and even, is $1 + n + \frac{1}{2}n \,.\, \overline{n-1} + \dots + n + 1 = (1+1)^n = 2^n$, and the number of even terms is clearly equal to the number of odd terms.

I shall call the algebra whose units are the even terms formed with n elementary units $\iota_1 \iota_2 \dots \iota_n$, the *n-way geometric algebra.* Thus quaternions are the *three-way* algebra. We may regard the units of quaternions as expressed in either of two ways. First, in terms of the elementary units $\iota_1 \iota_2 \iota_3$; they are then $(1, \iota_2\iota_3, \iota_3\iota_1, \iota_1\iota_2)$. Secondly, we may write k_1, k_2 for the terms $\iota_2\iota_3$, $\iota_3\iota_1$, and the system may then be written $(1, k_1, k_2, k_1 k_2)$. In this second form it is identical with the entire algebra of two elementary units, including both odd and even terms.

The five-way algebra depends upon the five terms k_1, k_2, k_3, k_4, k_5 and their products; the number of terms is sixteen. Now *we may obtain the whole of these sixteen terms by multiplying the quaternion set*

$$(1, k_1, k_2, k_1 k_2)$$

by this other quaternion set

$$(1, k_4 k_5, k_5 k_3, k_3 k_4).$$

For each of the sixteen products so obtained is a term of the even five-way algebra, and the products are all distinct. Moreover, *the two quaternion sets are commutative with one another.* For since the k multiply in the polar manner, we may regard them as elementary units for this purpose; now the terms in the second set are all even, and no term in one set has a factor common with any term in the other set.

In the language of Professor Peirce, then, the five-way algebra is a compound of two quaternion algebras, which do not in any way interfere, because the units of one are commutative in regard to those of the other. A quantity in the five-way algebra is in fact a quaternion $\omega + ix + jy + kz$, whose coefficients $\omega\, x\, y\, z$ are themselves quaternions of another set of units $(1, i_1, j_1, k_1)$, the i_1, j_1, k_1, being commutative with i, j, k.

I shall now extend this proposition, and shew that *the* $(2m+1)$-*way algebra is a compound of* m *quaternion algebras, the units of which are commutative with one another.* To this end let us write $p_0 = k_1 k_2$, and then

$$\begin{array}{ll} p_1 = k_1 k_2 k_6 k_7 = p_0 k_6 k_7 & q_1 = k_3 k_4 k_5 \\ p_2 = p_1 k_{10} k_{11} & q_2 = q_1 k_8 k_9 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ p_r = p_{r-1} k_{4r+2} k_{4r+3} & q_r = q_{r-1} k_{4r} k_{4r+1}. \end{array}$$

Consider now the quaternion sets

$$\begin{array}{l} 1, k_1, k_2, k_1 k_2 \\ 1, k_4 k_5, k_5 k_3, k_3 k_4 \\ 1, p_0 k_6, p_0 k_7, k_6 k_7 \\ 1, q_1 k_8, q_1 k_9, k_8 k_9 \\ 1, p_1 k_{10}, p_1 k_{11}, k_{10} k_{11} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 1, q_{r-1} k_{4r}, q_{r-1} k_{4r+1}, k_{4r} k_{4r+1} \\ 1, p_{r-1} k_{4r+2}, p_{r-1} k_{4r+3}, k_{4r+2} k_{4r+3} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

viz: a p-set and a q-set alternately. I say that if we consider the first m sets of this series, we shall find them to involve $2m+1$ of the k; that the products of m terms, one from each series, constitute 2^{2m} distinct terms, which are therefore identical with the terms of the $(2m+1)$-way algebra; and that the terms in any two sets are commutative with each other. The first two remarks are obvious on inspection; the last also is clear for the case of a p-set and a q-set, because the q-set is of even order in the k, and no factors are common to the two sets. It remains only to examine the case of two p-sets and of two q-sets. Consider the two p-sets

$$\begin{array}{l} 1, p_{r-1} k_{4r+2}, p_{r-1} k_{4r+3}, k_{4r+2} k_{4r+3}, \\ 1, p_{s-1} k_{4s+2}, p_{s-1} k_{4s+3}, k_{4s+2} k_{4s+3}, \end{array}$$

where $s > r$. All the terms of the first set are contained as factors in each of the terms $p_{s-1} k_{4s+2}, p_{s-1} k_{4s+3}$, which are of odd order in the k; consequently, the multiplication is scalar. The term $k_{4s+2} k_{4s+3}$ has no factor common with the first set, and being of even order is commutative in regard to it. Hence the two sets are commutative with one another. Next take the two q-sets

$$\begin{array}{l} 1, q_{r-1} k_{4r}, q_{r-1} k_{4r+1}, k_{4r} k_{4r+1}; \\ 1, q_{s-1} k_{4s}, q_{s-1} k_{4s+1}, k_{4s} k_{4s+1}. \end{array}$$

Here again all the terms of the first set are factors of $q_{s-1} k_{4s}$ and of $q_{s-1} k_{4s+1}$, and they have no factors in common with $k_{4s} k_{4s+1}$; since then all the terms are of even order in the k, the multiplication is scalar. The proposition is therefore proved.

We may set out a formal proof that the 2^{2m} products of m terms, one from each of the first m sets, are all *distinct*, as follows: Suppose this true for the first $m-1$ sets: that is to say, that no two of the products formed from them are either identical or such that their product is $\pm k_1 k_2 \ldots k_{2m-1}$. Let then a, b be two of these products; and let c, d be two terms of the next set. Then we have to prove that ac can neither be equal to $\pm bd$, nor such that the product $acbd$ is $\pm k_1 k_2 \ldots k_{2m-1} k_{2m} k_{2m+1}$. Now if $ac = \pm bd$, multiply both sides by bc; then $ab = \pm cd$. The product cd is one of the terms of the new set; it is either unity, or contains one or both of the new units k_{2m}, k_{2m+1}, so that it cannot be equal to ab. The product $abcd$ cannot be $\pm k_1 \ldots k_{2m+1}$ unless cd is $k_{2m} k_{2m+1}$ and ab is $k_1 k_2 \ldots k_{2m-1}$, which is contrary to the supposition. Hence if the products of the first $m-1$ sets are all distinct for the purposes of the $(2m-1)$-way algebra, the products of the first m sets will be all distinct for the purposes of the $(2m+1)$-way algebra. But it is easy to see that the products of the first two sets are distinct.

Algebras with an even number of units.

Every algebra with $2m$ units is related to the adjacent algebra with $2m-1$ units in precisely the same way as biquaternions are related to quaternions; namely, it is simply that adjacent algebra multiplied by the double algebra $(1, \omega)$ where ω is the product of all the $2m$ units. For clearly all the even terms of the $(2m-1)$-way algebra are also even terms of the $2m$-way algebra, and so also are their products by ω; but these are all distinct from one another, and consequently are *all* the even terms of the $2m$-way algebra.

The multiplication of ω with the k of the $(2m-1)$-way algebra is scalar, because the k are factors in the ω, and they are both even terms.

Hence the $2m$-way algebra is a product of the $(2m-1)$-way algebra with the double algebra $(1, \omega)$, the two sets of units being commutative with one another.