Alternative Path to Solve Einstein's Field Equations

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Abstract

A solution to Einstein's gravity field equations is derived directly from his 1916 paper, although it utilizes the conversion from Cartesian to polar co-ordinates, citing Schwarzschild's paper for its derivation.

Introduction

Einstein published his gravity field equations in his 1916 Ann. der Phys. paper. He suggested it would be difficult to determine their solution. Within a year Schwarzschild published a paper giving a solution to his field equations. Mathematicians found out that Schwarzschild introduced approximations to obtain his solution.

To overcome that, many went on to derive their own solutions. Their generally accepted format of their solution is:

$$ds^2 = dt^2\left(1 - \frac{\alpha}{r}\right) - \frac{dr^2}{\left(1-\frac{\alpha}{r}\right)} - (r^2 d\theta^2 + sin^2 d\theta d\phi^2) \qquad (1)$$

where s is the space-time co-ordinate and $\alpha = \frac{2GM}{c^2}$ is the Schwarzschild radius. The remaining terms are those customarily used.

The equation 1 solution, like Einstein's paper deriving them, are complex and difficult for most people to follow. The topic is left to experts. This presentation suggests it was not necessary for Einstein to derive his field equations. It goes directly from his work prior to his field equations, to the exact solution to the gravity effects he was describing earlier. It is the solution to his gravity theory without the need for his field equations.

Derivation

In his § 8 (of 22 §s), Einstein introduced the standard four dimension spatial tensor differential equation:

$$ds^2 = g_{\mu\nu} d_{x\mu} d_{x\nu} \qquad (2)$$

He repeated it in § 22. It uses the normalization of setting the speed of light, c = 1. His first derived field equations were his equation (47) in § 14. This presentation goes from equation (2) above, extracted from his § 8 to exact the exact solution.

Expanding $g_{\mu\nu}$ gives:

$$g_{\mu\nu} = \begin{matrix} g_{xx} & g_{xy} & g_{xz} & g_{xt} \\ g_{yx} & g_{yy} & g_{yz} & g_{yt} \\ g_{zx} & g_{zz} & g_{zz} & g_{zt} \\ g_{tx} & g_{ty} & g_{tz} & g_{tt} \end{matrix} \qquad (3)$$

Einstein interchangeably used the notation $x = 1, y = 2, z = 3$ *and* $t = 4$. The $\mu\mu$ terms are the μ term squared. Equation 3 can be re-written as:

$$g_{\mu\nu} = \begin{matrix} g_x^2 & g_{xy} & g_{xz} & g_{xt} \\ g_{yx} & g_y^2 & g_{yz} & g_{yt} \\ g_{zx} & g_{zy} & g_z^2 & g_{zt} \\ g_{tx} & g_{ty} & g_{tz} & g_t^2 \end{matrix} \qquad (4)$$

Adding the differential terms gives:

$$g_{\mu\nu}d_{\mu\nu} = \begin{matrix} g_x^2dx^2 & g_{xy}d_{xy} & g_{xz}d_{xz} & g_{xt}d_{xt} \\ g_{yx}d_{yx} & g_y^2d_y^2 & g_{yz}d_{yz} & g_{yt}d_{yt} \\ g_{zx}d_{zx} & g_{zy}d_{zy} & g_z^2d_z^2 & g_{zt}d_{zt} \\ g_{tx}d_{tx} & g_{ty}d_{ty} & g_{tz}d_{tz} & g_t^2d_t^2 \end{matrix} = ds^2. \qquad (5)$$

Calculating the individual $g_{\mu\nu}$, and $d_{x\mu}d_{x\nu}$ is difficult. However, gravity is spherically symmetric for a massive body, see figure 1. In that situation, $g_x = g_y = g_z = g_1$, making

$g_{\mu\mu} = g_{11} = g_x^2 = g_y^2 = g_z^2$. Equation (5) to be re-written as:

$$ds'^2 = \begin{matrix} g_{11}^2dx^2 & A & B & C \\ D & g_{22}^2d_y^2 & E & F \\ G & H & g_{33}^2d_z^2 & J \\ K & L & M & g_{44}d_t^2 \end{matrix} \qquad (6)$$

where *ds'* is the differential term in Cartesian co-ordinates. It is the same *ds* term used in the polar co-ordinate solution of equation (1). It is used only to show the format is different. All $\mu\nu$ components have been replaced by *A* to *M* respectively.

Einstein used the nomenclature that *x*, *y*, and *z* refer to the three orthogonal space dimensions and *t* is time. As such, the only two important dimensions are $x = g_1$ and $t = g_4$. In his § 22, Einstein derived:

$$g_{11} = -\left(1 + \frac{\alpha}{r}\right) \qquad (7)$$

That means that $g_{22} = g_{33} = -\left(1 + \frac{\alpha}{r}\right)$ (7a)

The question arises, “from where did Einstein get his α term?” He inferred it in his 1911 paper “On the Influence of Gravitation on the Propagation of Light. It was derived by using Newtonian gravity on packets of electromagnetic energy, i.e., photons, that he described as having mass in his 1905 paper “Does the Inertia of a Body Depend upon its Energy Content?”

Its origin is not otherwise referred to in his 1916 foundations paper. However, it is apparent from his work, and that of others who followed, that he was using the $\alpha = \frac{2GM}{c^2}$ used in equation (1). It adds to the confusion associated with his work.

He used the notation that $-1 = g_{11}dx_1^2$. Re-arranged that gives:

$$dx_1 = -\frac{1}{\sqrt{g_{11}}}. \qquad (8)$$

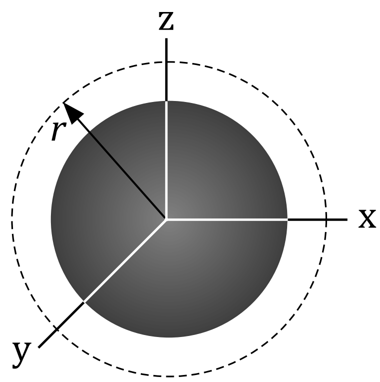

Figure 1 Schematic illustration of the symmetry of gravity associated with a massive object.

In the spherically symmetric situation of gravity associated with a massive object, x and r are interchangeable, see figure 1. So are their derivatives. Inserting equation (7) into equation (8) gives:

$$dx = \frac{1}{\left(1+\frac{\alpha}{2r}\right)} \tag{9}$$

when $\alpha \ll r$. Einstein went on to state "*it follows that, correct to a first order of small quantities,*

$$dx = 1 - \frac{\alpha}{2r} \text{ "} \qquad \text{(Einstein's equation 71)}$$

That is, $\frac{1}{(1+x)} \approx 1 - x$ when x << 1. It is an approximation that is only valid for r≫ α. Einstein regularly used that approximation in in his 1916 "Foundations" paper. As such, any exact solutions to his field equations will always be approximations.

Staying with his original solution of $dx = \frac{1}{\left(1+\frac{\alpha}{2r}\right)}$, gives:

$$dx^2 = \frac{1}{\left(1+\frac{\alpha}{r}\right)} \text{ when } \alpha \ll r. \tag{10}$$

Multiplying equations (7) and (10) gives:

$$g_{11}dx^2 = -\left(1+\frac{\alpha}{r}\right).\frac{1}{\left(1+\frac{\alpha}{r}\right)} = -1 \tag{11}$$

In a radially symmetric solution, his equation $g_{11} = g_{22} = g_{33} = -\left(1+\frac{\alpha}{r}\right)$. In a set gravitational field, the speed of light is constant. A change in length results in a negative inverse change in time. That gives:

$$g_{44} = \frac{1}{\left(1+\frac{\alpha}{r}\right)} \tag{12}$$

In his equation 70, Einstein approximated it to

$$g_{44} = 1 - \frac{\alpha}{r}$$

In the same manner, that gives $dt = \left(1+\frac{\alpha}{2r}\right)$and $dt^2 = \left(1+\frac{\alpha}{r}\right)$. That gives:

$$g_{tt}dt^2 = \frac{1}{\left(1+\frac{\alpha}{r}\right)} . \left(1+\frac{\alpha}{r}\right) = 1 \qquad (13)$$

Inserting equations (11) and (13) into equation (6) gives:

$$ds'^2 = \begin{matrix} -1 & A & B & C \\ D & -1 & E & F \\ G & H & -1 & J \\ K & L & M & +1 \end{matrix} \qquad (14)$$

Equation (14) informs us only that time is different from space. Beyond that, it is difficult to work out what is happening. That invokes what Einstein considered one of his greatest thoughts. An internal observer cannot tell the difference between free falling under gravity or being in a gravity free zone. Nor could an observer tell the difference between being at rest in a gravitational field or being accelerated in gravity free space.

In order to work out what is happening, it is necessary to fix one of them. Fixing the derivatives means the positions are fixed and we can determine the gravitational fields at that position. Fixing g is the equivalent of uniform acceleration. Fixing the derivatives means fixing a point in space where Newton's g has a fixed value. Fixing them at 1 allows the result to be multiplied by any value of Newton's g in future calculations.

That makes it apparent that Einstein's field equations deal with the difference between his gravity theory and Newtonian gravity. His calculations do not determine absolute gravity values.

Using equations (7) and (12), equation (14) becomes:

$$ds'^2 = \begin{matrix} -\left(1+\frac{\alpha}{r}\right) & A & B & C \\ D & -\left(1+\frac{\alpha}{r}\right) & E & F \\ G & H & -\left(1+\frac{\alpha}{r}\right) & J \\ K & L & M & \frac{1}{\left(1+\frac{\alpha}{r}\right)} \end{matrix} \qquad (15)$$

Converting from Cartesian to polar co-ordinates is virtually a look up equation. It was done by Schwarzschild in his 1916 paper. Such conversion gives:

$$ds^2 = \frac{dt^2}{\left(1+\frac{\alpha}{r}\right)} - dr^2\left(1+\frac{\alpha}{r}\right) - \left(r^2 d\theta^2 + sin^2 d\theta d\phi^2\right) \qquad (16)$$

The A to M values can be determined by calculating back from equation (16), if desired.

Equation (16) is the exact solution to Einstein's field equations. It differs from the accepted Schwarzschild solution because those who removed Schwarzschild's approximation did not remove the approximations Einstein made. Those approximations come in two forms.

His choice of tensors limited the accuracy of his work to second order tensors. That limited his whole study to approximations. It was good for r > ≈ 3α, a justifiable approximation.

The second approximations came through approximating $\frac{1}{\left(1+\frac{\alpha}{r}\right)} \approx . \left(1-\frac{\alpha}{r}\right)$. When $\frac{\alpha}{r} \approx 10^{-8}$, that is a valid approximation. It does not apply when r approaches α. Einstein mentioned his use of approximations several times. They can also be picked up by following his equations. Exact solutions to approximations always remain approximations.

Equation (16)'s ds^2 solution is the variation in gravitational fields between that predicted by Newtonian mechanics and that predicted by Einstein's theory. The total gravitational field strength is determined by replacing the normalized gravitational field strength by Newton's gravitational field strength of $g_N = \frac{GM}{r^2}$. That holds for any value of r and M. It gives the total gravity field strength under Einstein's theory, g_E as (Robinson 2021):

$$g_E = \frac{GM}{\left(1+\frac{\alpha}{r}\right)r^2} \tag{17}$$

Equation (17) shows that Einstein's gravity is a slight modification to Newton's theory. His use of approximations mean that calculations pertaining to extend his results are only valid for $r \gg \alpha$. It is well known, although derived again by Robinson (2021), that redshift $z = \frac{\alpha}{2r}$, giving:

$$g_E = \frac{GM}{(1+2z)r^2} \tag{18}$$

when $r \gg \alpha$.

Einstein's space–time distortion is photon redshift. It is still the cause of gravity, even when photon redshift is too small to measure. Equations (17) and (18) predict that gravity is weaker than Newtons' inverse square law.

Conclusion

This study has shown that equation 16, the exact solution to Einstein's gravity, could be determined without his field equations. Equations 17 and 18 are the complete gravity field equations for gravity under Einstein's approximations. They are much easier to use and provide exact solutions to the gravity field equations he derived. There is nothing in the above that wasn't available after 1916. As such there is no reason why others couldn't take the same approach. This suggests that excessive maths complexity has led to incorrect an understanding of what is otherwise a "relatively" simple topic.

Post script

This study has been extended to determine the property of nucleons that alter the electric permittivity of matter free space. The radial differential of that electric permittivity gives an equation in which Newton's inverse square law is a first approximation and Einstein's gravity is a second approximation. The exact solution, given in Robinson (2023) predicts the luminous torus shaped objects that have been detected at the centre of two galaxies. They are not black holes, in which Einstein never believed.

Equations (17) and (18) show that gravity is weaker than inverse square. That has significant consequences in cosmology.

When gravity is inverse square, an infinite steady state universe will collapse.

$g = \frac{4\pi r^3 \rho}{3r^2} \to \infty$ as $r \to \infty$. When gravity is less than inverse square, $g \to 0$ as $r \to \infty$.

An infinite steady state universe will not collapse! In his 1917 paper on Cosmological considerations, Einstein considered only a finite universe. Such a universe will collapse under any gravity origins.

As shown above, Einstein's calculations in his 1916 foundations paper showed that gravity was weaker than inverse square. Not understanding that because of Einstein's complex maths, cosmologists' attempts at fitting astronomers' observations to their theory has led to increasing complexities. Those complexities were resolved in Robinson (2023).

References:

Einstein A, 1905; Ann. der Phys., **18**, 639

Einstein A, 1911; Ann. der Phys., **35**, 898

Einstein A,1916; Ann. der Phys., **42**, 769

Einstein A, 1917; Sitzung. Preuss. Akad. der Wiss., 142

Robinson V N E, 2021; *J. Phys. Commun.* **5** 035013
https://doi.org/10.1088/2399-6528/abee2f

Robinson Vivian, 2023; The Common Sense Universe, Kindle, Chs 9 – 11.

Schwarzschild K, 1916; (English language version) https://arxiv.org/pdf/physics/9905030.pdf