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# Investigation of Electromagnetism in a Real Dirac Algebra

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*A Thesis Submitted to the  
University of Glasgow  
for the degree of  
Doctor of Philosophy*

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*If one is working from the point of view of getting beauty into one's equation, ...  
one is on a sure line of progress.*

P.A.M Dirac[1]

## Abstract

The primary aim of this thesis is to investigate the utility of a Clifford-Dirac algebra  $\mathcal{Cl}_{1,3}$  in describing relativistic electromagnetism. The motivation for choosing this algebra over other potential choices lies in the success of this algebra in describing relativistic quantum mechanics. The unit imaginary  $\mathbf{i}$  is excluded in this algebra because a purely real description is sought and the supposition that the element is not required in order to formulate a covariant electromagnetic theory is to be tested.

Square roots of the basis elements of the algebra are also investigated. An exhaustive computer search algorithm is developed to search for roots. This analysis provides general conditions for the formation of square roots where the basis elements have coefficients of equal magnitude. It is shown that such roots with two terms exist only for those elements which square to -1. No square roots are found with more than six terms. A simple algorithm is developed which enables the computational manipulation of Dirac-Clifford basis elements. A modified bubble-sort is used to perform multiplication of basis elements. This new algorithm is a reliable mechanism for performing multiplication.

Maxwell's equations are developed using the Clifford-Dirac algebra. The derivation of the complete set of field equations appear in a particularly compact and elegant form. A further motivation is to explore the potential for new kinds of wave functions within the algebra. The larger number of basis elements which square to -1 presents the opportunity to study wave equation using replacements for the complex imaginary. Finally, the aim of this thesis is to examine the behaviour of the electromagnetic field equations under relativistic transformations to determine whether or not the field equations and algebra form a relativistically covariant system.

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# List of Publications

S.J. Leary, J.G. Williamson, M.B. van der Mark, “*On Square Roots of the Hypercomplex Elements of the Dirac  $\gamma$ -Matrix Algebra*”, Reviewed internally at Philips Research Labs, Eindhoven (2005)

J.G. Williamson, M.B. van der Mark & S.J. Leary “*On the Nature of Photons*”, Reviewed internally at Philips Research Labs, Eindhoven (2006)



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# Chapter 1

## Introduction

Electromagnetism is concerned with the physics of electromagnetic fields. Electric and magnetic fields occur in nature and are generated by elementary particles. The term electromagnetism derives from the fact that electric and magnetic fields are closely intertwined and under many circumstances it is impossible to consider the two separately. For instance, a changing magnetic field gives rise to an electric field; this is the phenomenon of electromagnetic induction, which underlies the operation of electrical generators, induction motors, and transformers.

In the modern world applications of electromagnetic theory are readily apparent. All modern communications systems have utilised electromagnetic theory in their design and construction, regardless of whether the transmission medium is optical fibre or copper based. This is also the case for modern optical storage medium and certain types of laser medical applications such as microscopy.

The pursuit of more advanced field theories capable of better representing reality and providing better tools for describing such electromagnetic phenomena is the primary motivation for this research. In order to achieve an accurate description of what is real, an algebra is required which exactly describes the space and time of reality. As a consequence a particular four dimensional algebra is investigated in this thesis with a view to determining whether or not electromagnetism may be described as well as or better than alternative approaches. The use of this algebra as a tool compared to others is also examined.

## 1.1 History of Electromagnetism

The first accounts of magnetism can be traced back to a Greek shepherd known as Magnus, around 900 B.C, from the region now known as Magnesia. Although accounts of Magnus are difficult to corroborate, it is said Magnus noted that certain stones, now known as magnetite, attracted iron nails and the iron tip from his shepherd's staff. This material, Magnetite  $Fe_3O_4$ , is a natural magnet. Although Chinese fortune tellers used lumps of Magnetite to construct their fortune telling boards around 200 B.C, it was not until almost 1000 years after their initial discovery that the practical application of "Lodestones" in the construction of compasses was discovered. These compasses were used by early navigators to locate magnetic north.

In 1600 William Gilbert, court physician to Queen Elizabeth, published his book entitled "De Magnete" [2, 3]. This book contains a number of experimental results in magnetostatics & electrostatics and Gilbert clearly enjoys himself as he exposes the opinions of his contemporary theoreticians who had never themselves observed the phenomena they were seeking to elucidate[4]. Gilbert sought to understand how magnets acted upon each other without being in contact with electrified bodies and that both electric and magnetic forces appeared to act at a distance. Moreover he discovered that the the magnetic force was not a simple attractive force but was in fact a "coition" which involved a rotation. Gilbert was the first to put forth that the earth itself was a large magnet and used his knowledge in attempts to enhance the usefulness of the mariners compass.

Joseph Priestley was the first to suggest a mathematical basis for electrical phenomena. He postulated a correspondence between the inverse square law associated with gravitation and the action at a distance of electric and magnetic forces. Priestley's suggestion was investigated experimentally by Henry Cavendish who concluded that this was the case to within an experimental tolerance. However this work was never published by Cavendish and it remained unknown for a century before being compiled into a collected works publication by James Clerk Maxwell. The first public account of the inverse square law of these forces was given by Charles Augustin Coulomb and hence now bears his name.

In 1820 Hans Christian Oersted found that there was an effect on a magnetic compass needle when placed in proximity to a conducting wire connected to a voltaic battery. The magnetic action of the current appeared form a circular pattern around the current. This is the first account of a link between electricity and magnetism. Thus electromagnetism was born. The first mathematical explanation of electromagnetism was put forth by André Marie Ampère. Ampère developed a complete mathematical theory which made use of the idea of “current-elements”, which are short lengths of electric current. Ampère put forward the case for instantaneous action at a distance between electrical charges in a similar way to Newton’s theory of gravitation. This belief was about to be undermined by Michael Faraday.

Michael Faraday is generally regarded as the father of modern field theory\*. Although accounts[4] suggest that Faraday would have been less than pleased to be remembered as a theoretician. Theories, which he labelled “speculations” were in his view only useful in their utility in describing physical phenomena which he called “things themselves”. Faraday found that a magnet could be made to rotate continuously around a current-carrying conductor and demonstrated that a magnet could induce electrical current in a wire. Thus he was able to convert mechanical energy into electrical energy and construct the first dynamo. Lines of force were central to Faraday’s thinking and it was this that enabled him to deal with the effects of time. This eventually led to the field analogy with the Newtonian idea of instantaneous action at a distance being largely abandoned.

Perhaps the most important person in the history of electromagnetic theory is James Clerk Maxwell. One of Maxwell’s most important achievements was his extension and mathematical formulation of Faraday’s theories of electricity and magnetic lines of force. Analogy played a central role in Maxwell’s reasoning, as it did with Faraday. Both utilised it as a means of suggesting ways of exploring phenomena. However, where Faraday proceeded by experimentation, Maxwell utilised mathematical formalism. The first analogy used by Maxwell was that electric and magnetic phenomena were like those of the motion of an incompressible fluid. Within this model particles represented sources and sinks of fluid. The

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\*There is, regrettably, insufficient space available in this introduction and thesis to do full justice to the contribution to science put forth by Faraday. The interested reader is directed to the many biographies available.

incompressibility was equivalent to the inverse square law, because the velocity of the fluid issuing from a source would vary in this manner. In this analogy, lines of force were replaced by tubes of flux.

Maxwell's definitive paper on electricity and magnetism, *A Dynamical Theory of the Electromagnetic Field*[5], was published in 1864. In this paper he introduces the displacement current and argues that this is a real current as it concerns energy interchange. The most dramatic prediction, however, of Maxwell's theory of electromagnetism was the existence of electromagnetic waves moving at the speed of light, and the conclusion that light itself was just such a wave. This speculation challenged experimentalists to generate and detect electromagnetic radiation using some form of electrical apparatus. The first clearly successful attempt was by Heinrich Hertz in 1886. He used a high voltage induction coil to cause a spark discharge between two pieces of brass.

The influence of Maxwell in the century which followed is nothing short of remarkable. The formulation of the electromagnetic field equations and the theories of light were instrumental in the conception of the Michelson-Morley experiment[6] in 1887 and the eventual discovery of special relativity in 1905[7]. The photo-electric effect, quantum mechanics and LASER physics have all come about as a result of Maxwell's work.

In the latter half of the 20th century formulations of Maxwell's equations using advanced algebras including the algebra of forms[4] have led to more convenient formalisms which have more direct physical interpretations in terms of the equations and the quantities being described.

## 1.2 History of Algebras

The history of algebras presented here begins with an introduction to the history of numbers themselves and a description of a famous problem in mathematics known as the Pythagorean catastrophe[8, 9]. Negative numbers and zero were concepts yet to be imagined for the Pythagoreans. They had hoped that all geometric systems could be described using rational numbers i.e. numbers of the form  $a/b$  where  $a$  &  $b$  are positive integers. A simple example, however, can be used to show that this hope was in vain. Take a simple right angle triangle in Euclidean geometry where the sides are of length unity. The square of the length

of the hypotenuse is given by  $1^2 + 1^2 = 2$ [10]. This length is  $\sqrt{2}$  which has no expression in rational numbers[8]. This discovery eventually led to formulation of the real number system.

The word “Algebra” comes from a book written in Arabic which revolutionised the way mathematics was conducted. The book, entitled “Al-jabr w'al-mugabalah”, was written by Abu Ja'far Ben Musa also known as al-Khowarizmi around 825 AD[11]. This book is widely regarded as being the origin of algebra and the basis for the, eventual, wider acceptance of zero as a number throughout Europe. The first use of the word “algebra” in English was by the Welsh mathematician and textbook writer, Robert Recorde[12].

From around 1500 A.D, zero had been accepted by European philosophers and irrationals were used freely although people still worried about whether they were really numbers. Negative numbers were known but were not fully accepted. Complex numbers were as yet unimagined. Full acceptance of all components the modern number system did not come until the 19th century.

The earliest references to square roots of negative numbers exists in the work of the Greek mathematician and inventor Heron of Alexandria in the 1st century AD, when he considered the volume of an impossible frustum of a pyramid. They became more prominent when in the 16th century closed formulas for the roots of third and fourth degree polynomials were discovered. That these formulas sometimes required the manipulation of square roots of negative numbers was soon realised. This was unsettling as negative numbers themselves were not, at this time, considered to be on entirely firm ground. The term “imaginary” for these quantities was coined by René Descartes in the 17th century and was intended to be derogatory. The existence of complex numbers was not completely accepted until the geometrical interpretation in the form of the *complex plane* was presented by Caspar Wessel in 1797[8]. It was later rediscovered by Jean Robert Argand in 1806, John Warren in 1828 and by Carl Friedrich Gauss before 1831.

In the 19th century British mathematicians took the lead in the study of algebra. Attention turned to many “algebras”; that is various sorts of mathematical objects ( i.e. vectors, matrices, transformations, etc.) and various operations which could be carried out upon these objects. Thus the scope of “algebra” was expanded to the study of algebraic form and structure and was no longer limited to ordinary systems of numbers. By this time negative numbers were widely

accepted.

Perhaps the most significant breakthrough in mathematics is the development of non-commutative algebras. These are algebras in which significance is attached to the ordering of the operation of multiplication. The first example of such an algebra is the quaternions which Hamilton first wrote down on the 16th of October 1843 by scoring them onto the side of Brougham bridge in Dublin, less than a year before Grassmann's exterior algebra was published. As a progression from this, in 1878 William Kingdon Clifford published his paper entitled "Applications of Grassmann's extensive algebra" [13] in which a non-commutative geometric product is presented. Although Clifford's motivations appear to be purely academic his work was influenced by Riemann and Lobachevsky. This paper is the seminal paper for the class of algebras now regarded by modern mathematics as *Clifford algebras*.

In the early 20th century Paul Dirac, in an effort to find a first order form of the relativistic Schrödinger equation, constructed a non-commutative algebra which fulfilled specific criteria [14]. In finding his non-commuting 'spin' quantities, Dirac rediscovered an algebra with many of the same properties as the four dimensional Clifford algebras. Dirac appears to have been completely unaware of Clifford's work. As a consequence, algebras with properties common to both are sometimes referred to as Dirac-Clifford algebras [8].

### 1.3 Motivation & Aims

The primary aim of this thesis is to investigate the utility of the Clifford algebra  $\mathcal{C}\ell_{1,3}$  in describing relativistic electromagnetism. The motivation for choosing this algebra over other potential choices lies in the success of this algebra in describing relativistic quantum mechanics. The algebra presented in this thesis will be shown have an isomorphism with a particular Dirac algebra. The Dirac electron theory is widely accepted to be the most precise available quantum theory of a single particle [15, 14].

The approach presented in this thesis differs from that of Dirac [14] in that although the basis elements are isomorphic, the interior and exterior products are treated separately and form part of a four dimensional geometric product. Further, the unit imaginary  $i$  is explicitly excluded from this algebra whereas

it is present in a general Dirac algebra. The motivation for excluding complex numbers is twofold. The first is because a purely real description is sought. The second is to test the supposition that the element is not required in order to formulate a covariant electromagnetic theory.

The properties of the algebra will be examined in detail. This is primarily to give the reader a complete description of the algebra and provide a comparison with other, perhaps more familiar, algebras. The motivations for choosing the particular metric will be explored. Newly discovered properties of the algebra will also be presented[16].

To develop computer software and algorithms which permit the manipulation of the presented algebra is also an aim of this thesis. The motivation for developing a software based system is due to the tediousness involved in calculating the large number of terms which appear for products in four dimensional algebras. Mistakes are easily made by humans performing such calculations, and one error in the signs can lead to an incorrect and unusable result.

A further aim is to examine the potential for new kinds of wave functions within the algebra. The motivation for this lies in the large number of basis elements which square to -1. This presents the opportunity to study the combinatorics of these elements in a wave equation using custom developed software.

The final aim of this thesis is to examine the behaviour of the electromagnetic field equations under relativistic transformations. The motivation for this examination is to determine whether or not the field equations and algebra form a relativistically covariant system. i.e. whether or not the system varies correctly when these transformations are applied.

## 1.4 Thesis Outline

This thesis is organised as follows:

The algebra which will be used for the investigation is a Dirac-Clifford algebra and will be presented in Chapter 2. This algebra will be presented from the Cliffordian point of view. A general overview of Clifford algebras is used as a means of introducing the full four dimensional algebra. Consequently the algebra is put forward in a three dimensional Euclidean space, before the progression to four dimensions is made. The motivation for utilising natural units is explored

and the means by which translation between natural and S.I units is presented. The properties of the four dimensional algebra will then be explored in depth. The metric tensor, the differential operator, the concepts of parallelness and perpendicularity will be examined. In addition, conjugates, duality operators and closed groups within the algebra will be investigated and analysed.

In Chapter 2 the definitions and properties of the algebra will be examined. Natural units, quaternions, conjugates and duality operations will be presented and discussed. The  $3 \rightarrow 4$  space projection used by Gull *et al* [17] will be examined using computational techniques (Chapter 3). This will be utilised as a basis for an investigation of the handedness of rotations in spaces with different (metric) signatures[18]. The concept of “parallelness” in four-space with a sixteen element algebra will also be explored. Furthermore sub-groups of this algebra which are isomorphic to the Lorentz and Quaternion groups will be identified. The properties of the Poincaré group will be examined.

Square roots quantities of elements within the algebra[16] will be examined using software and an exhaustive computer search algorithm. This algorithm is developed and presented in Chapter 3. A new test procedure is developed to search for roots. The analysis provides general conditions for the formation of square roots where the basis elements have coefficients of equal magnitude. Such roots with two terms exist only for those elements which square to -1. Four term roots are found for all basis elements. No square roots are found with more than six terms.

In Chapter 3 the development, usage and application of algebraic software will be presented. A review of some pre-existing packages capable of manipulating geometric algebras will be conducted and justification for the development of custom software put forth. A simple algorithm which enables the manipulation of Dirac-Clifford basis elements will also be demonstrated. A modified bubble-sort is used to perform multiplication of basis elements. This algorithm will be shown to be a clean and reliable mechanism for performing such multiplications.

The aim of Chapter 4 is to develop the familiar electromagnetic field equations. Maxwell’s equations will be developed in this chapter using the spacetime algebra presented in Chapter 2. The aim being to present the derivation of the field equations and show that the complete set of these equations appear in a particularly compact and elegant form. Once the field equations are introduced



and analysed, solutions to these equations will be derived. These solutions will take the form of travelling waves using elements of the algebra instead of the unit imaginary. The various polarisations of these solutions will be investigated; linear superposition along with normalisation steps will also be demonstrated.

An additional description of transverse waves will also be presented in Chapter 4. This description is a self-referencing wave function which will only propagate transverse combinations of electric and magnetic fields along the Poynting vector. This wave function will not propagate electric and magnetic fields individually.

The gauge degree of freedom will be presented in Chapter 4. This degree of freedom will be used to show how setting various portions which contribute to gauge simplifies the pursuit of solutions. Several such simplifications will be introduced and correspondences made with known gauge conditions.

Chapter 5 is concerned with the Lorentz transformations and covariance within the algebra presented in Chapter 2. The motivation for exploring these transformations is to check and ensure that the field equations transform covariantly. The Lorentz transformation will be introduced in this chapter and will be presented initially as an extension of the classical Galilean transform[19, 20]. The aim of this chapter is to provide a detailed analysis of Lorentz transformations and covariance within the presented algebra. Attention will be drawn to the way in which quantities and elements of the algebra behave under Lorentz transformations. The covariance of the field wave function solutions presented in Chapter 4 will be demonstrated. Particular attention is paid to the energies of these wave functions under transformation and comparisons will be drawn with experiment. Discrepancies will be discussed and resolved. Finally, the covariance of energy-momentum quantities using this algebra will be examined and discussed. The resulting form of the energy-momentum tensor will be examined in the context of the work of Boyer[21] and Rohrlich[22, 23].

## 1.5 Achievements & Publishable Results

The main achievements of the work presented here are as follows: -

- A fresh objective examination of the properties of the Clifford Algebra  $\mathcal{Cl}_{1,3}$  has been conducted. Relationships with other algebras have been explored and the isomorphisms highlighted.
- A new software technique has been developed for manipulating Clifford algebras. This has enabled the exhaustive examination of various problem-spaces and allowed patterns to be identified in the behaviour of the basis elements. A structured, object-oriented approach has produced a stable and extensible system.
- A new technique for determining roots of basis elements has been developed and this has been used to find conditions for the formation of square root quantities with coefficients of equal magnitude. This has been extended to form generalised expressions for such roots. This technique utilises the aforementioned software techniques.
- A new set of mathematically curious wave functions have been devised. These functions have been shown to be equivalent to the standard traveling wave functions for cases where the propagation operator  $\phi_{prop}$  and the fields meet certain conditions. For other cases the mathematics behind the equations do not simplify to form a travelling wave solution.
- A fresh investigation of Lorentz covariance in the Clifford algebra, making use once again of the new software, has resulted in a complete description of the problem. The rigorous method applied shows the behaviour of each element of the algebra under a Lorentz transformation and this has allowed a commutation-separation technique to be developed which simplifies the calculation of Lorentz transformations within this algebra. These techniques have also been extended to look for and find general conditions for covariant & invariant derivatives, and Lorentz invariants themselves.

Within the work presented in this thesis there are several sections which either have been, or could be, adapted to paper form and published in the literature.

Amongst these, the work presented in Section 2.11, “The Roots of Basis Elements” has been adapted to a paper entitled “On The Square Root Quantities of the Hypercomplex Elements of the Dirac  $\gamma$ -Matrix Algebra”. Similarly, the work presented in Section 4.6.5, “New Wave Functions” has been incorporated in a larger paper on electromagnetism entitled “On the Nature of Photons”. Both of these papers being targeted for submitted to J. Phys. A in conjunction with M.B van der Mark and J.G Williamson, with the author carrying first and third authorship on these publications respectively.

# Chapter 2

## Spacetime Algebra

### 2.1 Introduction

An algebra is a mathematical system for which the rules of addition, multiplication and multiplication by a scalar are defined over a set of elements[24, 25]. The vision of representing reality has led philosophers, theorists, mathematicians and the like to the development of algebras which are capable of acting as a vessel for the exploration of theories new and old. This chapter will introduce and detail the mechanics of the particular four dimensional algebra, the properties of which will be examined throughout this thesis. That this algebra is a purely real Clifford-Dirac algebra with Minkowski-Lorentz-Poincaré properties will be demonstrated (See section 2.9, “Properties of the Spacetime Algebra”). This algebra is often referred to in the literature as the “Spacetime algebra”[17, 26, 27, 28, 29, 18].

This chapter will begin with an introduction to Clifford algebras and the manner in which geometrical elements are represented by them. Although this is mainly a review chapter it is essentially an original re-examination of the field of Clifford algebras. Specifically sections 2.9 & 2.11 contain original research, sections 2.7 & 2.8 are original re-examination sections and the rest of the chapter is mainly review material.

The introduction of Clifford algebras will be followed by a presentation and discussion of, the salient properties of Clifford algebras. Some of the forms presented in Clifford algebras may be unfamiliar even to those well versed in other algebras such as the algebra of forms[4]. This chapter will introduce and discuss

how these algebras are capable of naturally representing geometric objects, for instance a directed plane, which are orientable in space (and time), and how this orientation is reflected in the algebra. A connection will be made with algebras which are perhaps more familiar, such as the conventional relativistic algebra[30].

The detailed behaviour of the algebra will also be investigated later in this chapter. How the concepts of natural units, quaternions, conjugates and duality operations are represented and applied to the Spacetime algebra will be presented and discussed. The 3→4 space projection used by Gull *et al* [17] will be examined using computational techniques (Chapter 3). This will be utilised as a basis for an investigation of the handedness of rotations in spaces with different (metric) signatures[18]. The algebra which will be presented here is that used by Gull *et al*[17, 31], and is also discussed by Lounesto[27] and R. Penrose[8]. As a consequence, their work is referred to, as appropriate, throughout this chapter and thesis.

## 2.2 Clifford Algebras

A defining characteristic which distinguishes Clifford algebras (and the Spacetime algebra, Section 2.6) from other algebras is the way in which vectors and other objects multiply with each other\*. Although these “Clifford spaces” accommodate vectors, it is an insufficient description to refer to them as “vector spaces”, therefore they will not be labelled as such. The term “vector” is reserved for describing a directed-line element and when referring to the work of others.

The Clifford space defined here is a linear space with an orthogonal basis. The addition of two row vectors as  $\vec{a} + \vec{b}$  and the scalar multiplication of vector  $\vec{a}$  by a scalar  $\lambda$ , as  $\lambda\vec{a}$  must be defined. Within this scope there is no essential difference between a linear “Clifford space” and a conventional linear vector space.

As mentioned earlier the differences between a vector space and a “Clifford space” become apparent when one engages in the multiplication of vectors. These differences arise because of the way that products[30, 17, 32, 27] are treated.

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\*For an excellent introduction to Clifford and Spacetime algebras see “Imaginary Numbers are Not Real” by Gull *et al*[17].

## 2.3 The Clifford Product

The Clifford product will be introduced in this section. For a short summary of the products used in this thesis see Appendix D. The Clifford or Geometric product is defined as[27, 32, 17]: -

$$AB = A \cdot B + A \wedge B \quad (1)$$

Which is a sum of the symmetric: -

$$A \cdot B = \frac{1}{2} (AB + BA) \quad (\textit{Symmetric}) \quad (2)$$

and anti-symmetric portions of the Clifford product in Eq. 1: -

$$A \wedge B = \frac{1}{2} (AB - BA) \quad (\textit{Anti-Symmetric}) \quad (3)$$

For the simple case of two row vectors; a correspondence can be made between the *Symmetric* and the *inner product*. i.e. : -

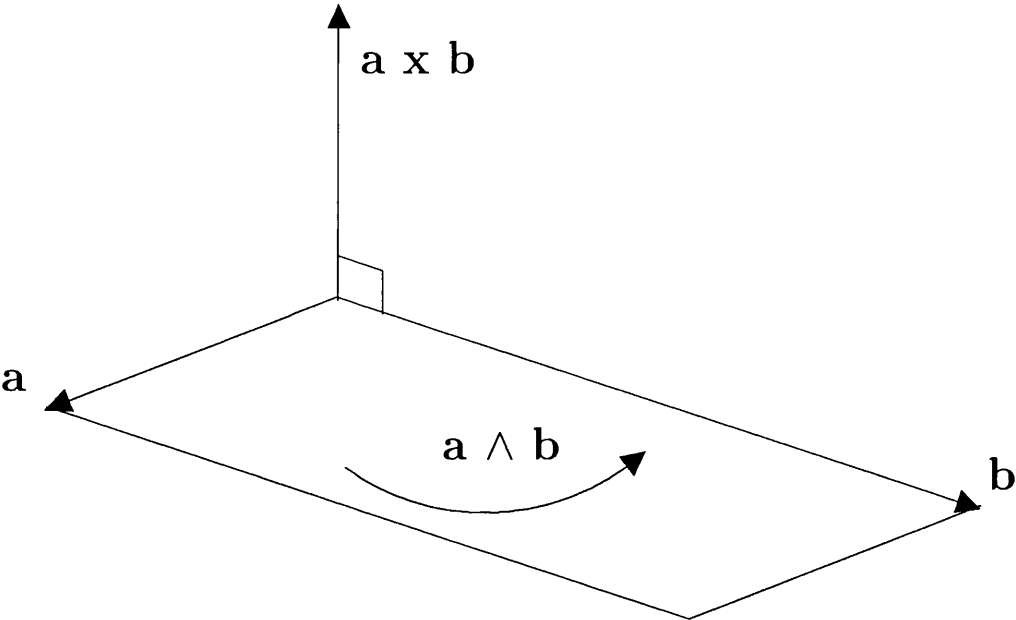
$$\vec{a} \cdot \vec{b} = a_i b_i, \quad i = \{1 \dots 3\} \quad (4)$$

where a summation convention over cyclic permutations of  $i$  is used in this case.\* The over-arrow notation denotes a row vector. This product is defined as the inner product and is applicable only to column and row vectors in this thesis. This will be discussed in more detail later.

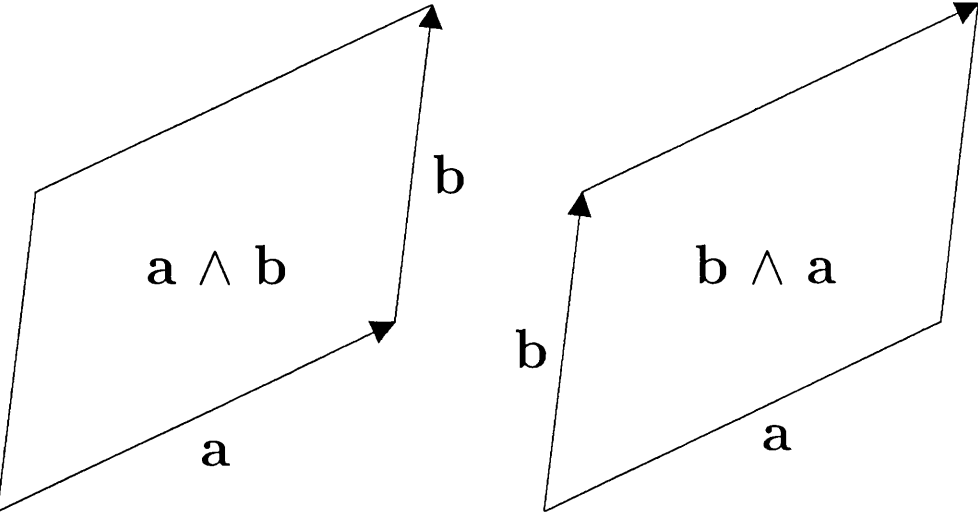
The use of the “dot” notation to denote the symmetric product, rather than being a strict inner product, may be confusing. This is, however, standard usage in the Clifford algebra literature[17, 27, 32, 33, 29, 34]. For the simple case of two (Clifford) vectors  $A$  and  $B$ , this is the sum of a scalar ( $A \cdot B$ ) and a bivector ( $A \wedge B$ ). This can be demonstrated by introducing a 3-vector with Clifford elements to represent the geometry (represented by the  $h$ 's below which are introduced and explained in the next section): -

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\*Note that in throughout this thesis a summation convention is not used unless explicitly stated.



**Figure 2.1.** Figure showing the difference between a conventional vector product in which the resultant is a vector (polar or axial) and the Clifford product two vectors.



**Figure 2.2.** Figure showing the wedge product which is a property of the plane defined by  $a$  &  $b$ . Here  $a \wedge b \neq b \wedge a$  and  $a \wedge b = -b \wedge a$  due to the change in direction implied by the differences in ordering.

$$A = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \vec{A} \quad (5)$$

The reversal of column and row vector in Eq. 5 implies an *inner product*. This notation is explained in greater detail in Section 2.6.2. The vector product correspondence ( $\vec{a} \times \vec{b}$ ), however, is not so straightforward. In a conventional linear vector space, there are many ways of treating this product (for three dimensional systems and higher); the most simple non-trivial of which is the projection of  $\vec{a} \times \vec{b}$  onto a third vector which is mutually perpendicular to  $\vec{a}$  and  $\vec{b}$  (i.e.  $\hat{x} \times \hat{y} = \hat{z}$ , See Figure 2.1). There are many cases where this treatment is insufficient. An example of this is where one wishes to represent a turning vector (e.g. a torque). In these cases, axial vectors may be used to replace the polar vector projection. This notation is an excellent representation for the case of three dimensions. When one tries to utilise the same technique in four dimensions, there exists an infinite plane of vectors, all of which are perpendicular to the two multiplying vectors. There is therefore an infinite number of choices for the projected vector, whether polar or axial in form. Whilst it is possible, in some cases, to project time out of the system to limit this choice, there are consequences in relativistic systems (See Chapter 5, “Lorentz Transformations”).

Such projections are not made in Clifford spaces and therein lies the essential difference between this space and a linear vector space. In a Clifford algebra, the “cross” product of two non-parallel vectors is preserved as a directed plane element, referred to as a bivector. This bivector is of a different grade to the two constituent vectors (analogous, but not isomorphic, to a 2-form. See Section 2.6.3, “Exterior Algebra Relationships”). It is often the case, in this thesis, that the elements which represent the directed plane elements (amongst other possibilities) are separated from the row vector representing the arguments of the quantity in the notation presented here. In these cases the overall quantity is presented as the inner product of the standard Heaviside-Gibbs representation with a column vector representing the geometric elements.



The product of two Clifford 3-vectors,  $A$  &  $B$  is : -

$$AB = \vec{A} \cdot \vec{B} + \begin{pmatrix} h_{23} \\ h_{31} \\ h_{12} \end{pmatrix} (\vec{A} \times \vec{B}) \quad (6)$$

$$A \cdot B = \vec{A} \cdot \vec{B} \quad (\text{Scalar}) \quad (7)$$

$$A \wedge B = \begin{pmatrix} h_{23} \\ h_{31} \\ h_{12} \end{pmatrix} (\vec{A} \times \vec{B}) \quad (\text{Bivector}) \quad (8)$$

The different forms of the scalar and bivector terms beg a crucial question regarding the addition rules of bivectors and scalars. According to Gull *et al*[17] many physicists require help in resolving this problem conceptually. Adding together a scalar and a bivector may seem amiss because they are different types of quantities. However, addition of this nature is precisely what is done in the conventional complex number system. i.e: -

$$Z = X + iY \quad (9)$$

where the real and imaginary portions are added together to form a complete object. Just as for the complex case, the scalar and bivector portions are linearly independent of each other. At this point it should be noted that in terms of the geometry, the scalar portion of  $AB$  corresponds to the product of those parts of  $A$  and  $B$  which are parallel; whereas the bivector portion is constructed from the perpendicular parts of  $A$  and  $B$ . As problems in electromagnetism are approached, one finds that being able to describe many different types of quantities within a single equation to be of enormous benefit (See Section 4.3, “The Maxwell Equations”). These compound objects must scale linearly: -

$$\lambda AB = \lambda A \cdot B + \lambda A \wedge B \quad (10)$$

The resultant form of the contracted portion of this product is not necessarily a simple scalar. The “dot” symbol used to represent the symmetric portion of the product is, therefore, not a strict “scalar” product and there is not always a correspondence between the symmetric portion of the product and the inner

product (See Appendix D for an example). Objects of higher dimensionality may be formed through this product in more complex scenarios. The general rule observed here is that the “dot” multiplication of two objects implies the symmetric portion of the Clifford product except in the case where one or more of the objects have an over-arrow. In which case an inner product is equivalent to the symmetric product and is therefore implied as no geometric basis elements are present.

## 2.4 Basis Elements

In this section the Clifford elements representing  $\mathcal{Cl}_{3,0}$  will be introduced. The elements representative of  $\mathcal{Cl}_{1,3}$  will be introduced in Section 2.6.

$$\hat{x} = h_1 \quad (11)$$

$$\hat{y} = h_2 \quad (12)$$

$$\hat{z} = h_3 \quad (13)$$

These elements (and the set from  $\mathcal{Cl}_{1,3}$  introduced later in this chapter) will be referred to throughout this thesis as vector basis elements\*. The basis set  $\{h_i\}$  has the property: -

$$h_i^2 = 1 \quad (14)$$

The commutation relations are such that the exchange of two non-like vector elements incurs a change in the overall sign of the Clifford product: -

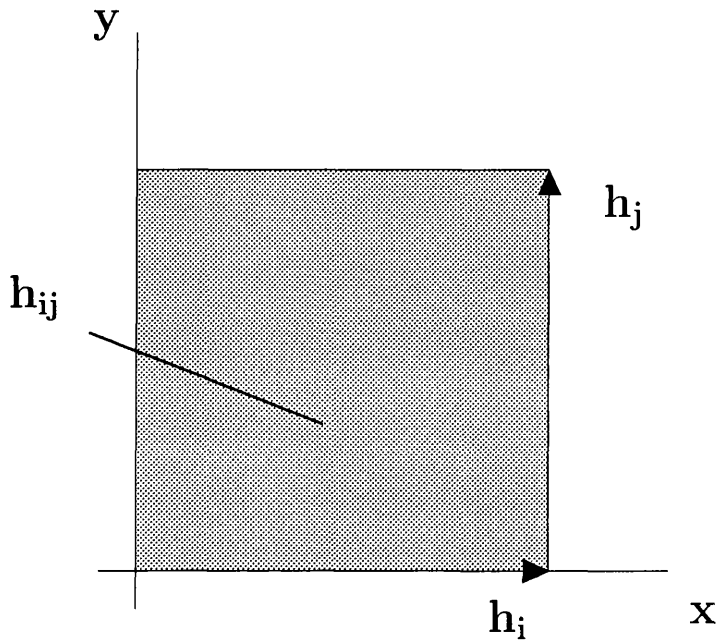
$$h_1 h_2 = -h_2 h_1 \quad (15)$$

$$h_1 h_2 h_3 = -h_1 h_3 h_2 \quad (16)$$

These elements form the generator set  $\{h_i\}$  which form the building blocks

---

\*Note that some authors[35] distinguish between a set of “M” vector basis elements and their dual set of “N” vector basis elements



**Figure 2.3.** Example of the product of two basis vector elements. The Clifford product of  $h_i$  into  $h_j$  is shown here as  $h_{ij}$ . The new object is a bivector and is an orientable plane element, the argument of which corresponds to the area of the unit bivector.

with which all the other kinds of objects (bivector, trivectors) are constructed. For example, a bivector may be constructed from a Clifford product of two of unit vectors: -

$$h_{ij} = h_i \wedge h_j + h_i \cdot h_j = h_i \wedge h_j + 0 \quad (17)$$

The commutation relations presented above imply that the symmetric portion of the product for non-like elements is: -

$$h_1 \cdot h_2 = 0 \quad (18)$$

This ordered bivector  $h_i h_j$  is denoted with Latin subscripts for simplicity i.e.  $h_{ij}$ . This is, however, not a summation convention over the indices. Instead each subscript denotes either a vector element (i.e.  $h_1$ ) or a normalised sum of vector elements (i.e.  $\frac{1}{\sqrt{2}}(h_1 + h_2)$ ) where each subscript combination is independent of

all others. This is a bivector basis element and is analogous, although not isomorphic, to the 2-form in the “Algebra of Forms”[4, 36, 37]. As the “Algebra of Forms” is an exterior algebra, whereas the algebra presented here is a geometric algebra containing symmetric and anti-symmetric products (for which a correspondence with the interior and exterior products may be made depending on the individual terms involved in any particular product), these elements will be referred to as bivectors rather than 2-forms throughout this thesis. Note that these bivector elements are neither vectors or matrices but are discrete elements themselves. For more on notation see Section 2.6.2, “The Base Vector Set & Metric Tensor”.

As the properties of this algebra are examined, one finds that for a three dimensional space there are three bivectors each of which are linearly independent of the three unit vectors. These three bivectors may be represented as: -

$$h_{23} = h_2 \wedge h_3 = -h_{32} \quad (19)$$

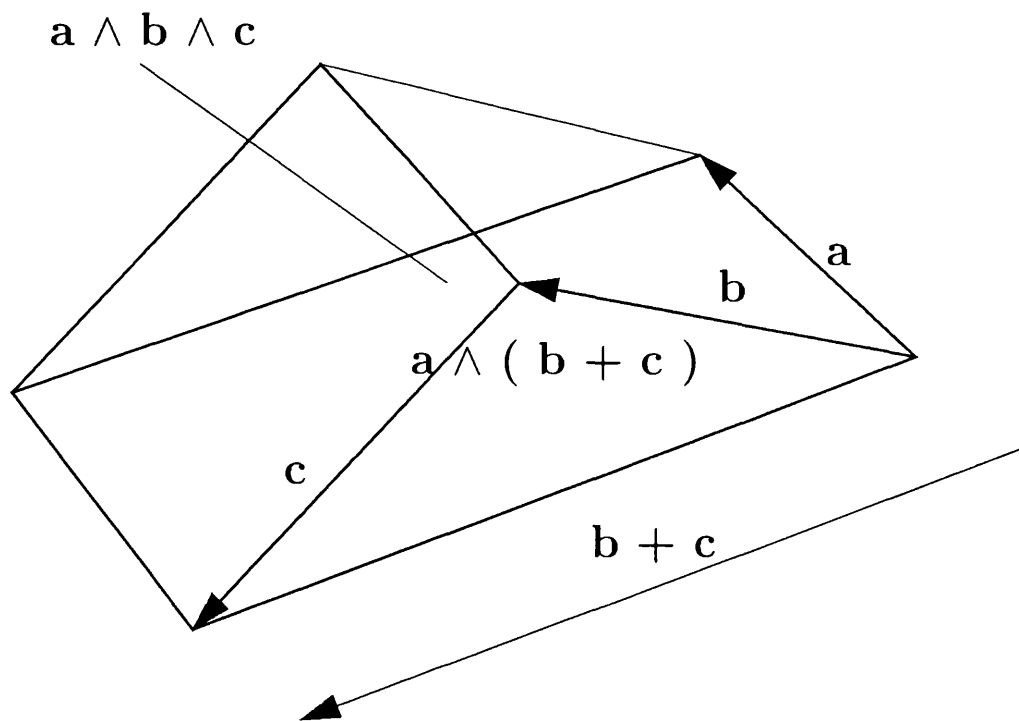
$$h_{31} = h_3 \wedge h_1 = -h_{13} \quad (20)$$

$$h_{12} = h_1 \wedge h_2 = -h_{21} \quad (21)$$

Although these bivectors may be projected onto vectors, they are themselves linearly independent entities. There is an additional object which spans all three dimensions. In the same way that a bivector is analogous to a directed plane element, there is an object which is analogous to a directed volume element (See Figure 2.4 for an example of a directed volume element). The object is referred to as a trivector, is denoted  $h_{123}$  and can be represented algebraically as the Clifford product of all three vector basis elements in  $\mathcal{Cl}_{3,0}$ . This object is analogous to a spatial 3-form object in the “Algebra of Forms”. Once again, as a consequence of the differences in the algebra, these objects will be referred to as trivectors in this thesis to prevent confusion. This object is defined as: -

$$h_{123} = h_1 \wedge h_2 \wedge h_3 + h_1 \cdot h_2 \cdot h_3 = h_1 \wedge h_2 \wedge h_3 + 0 \quad (22)$$

or as the Clifford product of a bivector and a “parallel” vector as follows: -



**Figure 2.4.** This figure shows directed line elements (vectors), directed plane elements (bivectors) and a single directed volume element (trivector). Here the volume element is directed inward.

$$h_{123} = h_1 \wedge h_{23} \quad (23)$$

$$= h_2 \wedge h_{31} \quad (24)$$

$$= h_3 \wedge h_{12} \quad (25)$$

Just as in the case for the bivectors, the argument of the trivector is independent of the specific shape, but the sign depends on the ordering. All even permutations of  $h_{123}$  are positive (as shown in Eq. (23), (24) & (25) ) and all odd permutations are negative.

## 2.5 Natural Units

Throughout this thesis, unless otherwise specified, the units of all equations and expressions are given in natural units. In the natural unit system the Planck quantities (those generated by algebraic combinations of  $G$ ,  $c$ ,  $\epsilon$  and  $\hbar$ ) are set to 1. This allows the manipulation of the physical quantities without the necessity of tracking the various prefactors, thus narrowing the scope for errors and reducing the overall time required for calculations.

In order to calculate the value of an expression in S.I. units a method is required for inserting appropriate values in place of the natural quantities in the terms. A conversion table is utilised for this purpose[38] (Table 2.1).

Units Conversion Table			
Natural	S.I.	Natural	S.I.
$t$	$ct$	$\vec{x}$	$\vec{x}$
$\partial_0$	$\partial_0/c$	$\vec{\partial}$	$\vec{\partial}$
$A_0$	$\Phi$	$\vec{A}$	$\vec{A}/c$
$\vec{E}$	$\vec{E}$	$\vec{B}$	$c\vec{B}$
$J_0$	$\rho/\epsilon_0$	$\vec{J}$	$\vec{j}/(\epsilon_0 c)$
$U$	$u/\epsilon_0$	$\vec{g}$	$c\vec{g}/\epsilon_0$
$\vec{L}$	$c\vec{L}/\epsilon_0$	$\vec{S}$	$\vec{S}/(\epsilon_0 c)$
$f_0$	$p/(\epsilon_0 c)$	$\vec{f}$	$\vec{f}/\epsilon_0$
$\mu_0 = 1/(\epsilon_0 c^2) = 4\pi \times 10^{-7} \text{ Js}^2/(\text{C}^2 \text{m})$ $c = 2.9979245 \times 10^8 \text{ m/s}$ $\hbar = 1.05457266(63) \times 10^{-34} \text{ Js}$ $\alpha = e^2/(4\pi\epsilon_0\hbar c) = 7.29735308(33) \times 10^{-3}$			

**Table 2.1.** This table is a standard set of conversions for S.I. to and from natural units

The relevance of this table is that it allows the reader to convert most conceivable calculated quantities from natural units to S.I and back. It is provided here in order to give a complete picture of the relationship between the equations presented in this thesis and those found in standard texts. Thus enabling the practical application of these equations. An example will now be presented. The conversions shown in Table 2.1 may be utilised in the following way: an

occurrence of a quantity from one side of the table should be *replaced in its entirety* by the value from the opposite column. Although physical interpretation is avoided in this chapter, some well known electromagnetic equations will be used to demonstrate these conversions.

$$U = \frac{1}{2} \left( \vec{E}^2 + \vec{B}^2 \right) \quad (\text{Natural}) \quad (26)$$

$$\frac{u}{\varepsilon_0} = \frac{1}{2} \left( \vec{E}^2 + c^2 \vec{B}^2 \right) \quad (\text{S.I.}) \quad (27)$$

In Eq. (26) & (27) above, one can see how the table functions on a real equation. The energy  $U$  is replaced by  $u/\varepsilon_0$  in Eq. (27); similarly,  $\vec{B}$  has been replaced by  $c\vec{B}$ . The value for  $\vec{E}$  has remained unaltered because the entries are equivalent in the tables. The transformed equation is now in the appropriate S.I. units.

## 2.6 The Spacetime Algebra

### 2.6.1 Introduction

In this section the particular algebra which will be utilised throughout the remainder of this thesis will be introduced and discussed. This algebra has a greater number of dimensions (four) than those introduced in previous sections.,

This algebra is a Clifford algebra. However, many of the properties differ from the three dimensional case presented earlier. One such property is the metric. As a consequence a different notation is utilised to prevent confusion. Only where explicitly stated is the Einstein summation convention, with Greek subscript indices,  $e_\mu$  for four-space is used. This is usually denoted with “(Summation)” to the right of the equation. Otherwise each subscript represents either a vector element (i.e.  $e_1$ ) or normalised sum of vector elements (i.e.  $\frac{1}{\sqrt{2}}(e_1 + e_2)$ ) where each subscript combination is independent of all others. This four dimensional algebra will be presented and the differences between it and the three dimensional algebra presented earlier will be put forth.

The properties of the metric tensor will be introduced and discussed in this section and will be accompanied with an illustration of some of its features.

This will be done by demonstrating and discussing the properties of the base vector set under multiplication. In addition, the four-vector will be introduced and discussed.

The full basis element set of 16 elements will be introduced and how these objects are formed through products of the vector basis will be shown. Finally, the quotient of a four-vector will be given and utilised to introduce the vector differential operator. The various properties of this differential operator will be discussed and compared to the work of others.

### 2.6.2 The Base Vector Set & Metric Tensor

The full algebra has a four dimensional basis set  $\{e_\mu\}$ . All the units are natural (See Section 2.5, “Natural Units”.) unless otherwise stated. Latin subscripts are used here when distinguishing the spatial components  $\{e_i, e_j, e_k\}$  from the time component  $e_0$ . Spatial position variables are denoted  $x_i$ . The properties of these basis elements are such that the following relationships are satisfied: -

$$e_0^2 = -e_i^2 = 1 \quad , \quad i = \{1 \dots 3\} \quad (28)$$

This configuration forms a Minkowski space[39, 40, 41, 42, 43]; this is the motivation for choosing different signatures for the temporal and spatial dimensions (See Section 2.9.2, “Minkowski Spacetime”). Strictly, the Minkowski space defined here is  $M^4$ , which is close to  $R^4$ , but endowed with the “pseudo-Riemannian” or “Lorentzian” metric[41] (See also R. Penrose[8] page 319). This particular space is sometimes denoted  $R^{1,3}$  to emphasise the metric signature. The anti-commutator is denoted as: -

$$\{e_\mu, e_\nu\} = e_{\mu\nu} + e_{\nu\mu} = 2g_{\mu\nu}\mathbb{1} \quad (29)$$

The metric tensor  $g_{\mu\nu}$  of the four dimensional algebra can be determined from the product  $e_\nu e_\mu$ : -



$$g_{\mu\nu} = e_\mu e_\nu = \begin{bmatrix} 1 & e_{01} & e_{02} & e_{03} \\ -e_{01} & -1 & e_{12} & -e_{31} \\ -e_{02} & -e_{12} & -1 & e_{23} \\ -e_{03} & e_{31} & -e_{23} & -1 \end{bmatrix} \quad \mu = \{0 \dots 3\}, \nu = \{0 \dots 3\} \quad (30)$$

The diagonal of this matrix forms the metric. This matrix contains scalar quantities (diagonal) and bivector quantities ( $e_{01}$ ,  $e_{23}$  etc.). Once again these bivectors are neither matrices, forms or vectors but are distinct elements themselves. An analogy with 2-form elements may be made, as both have the geometrical interpretation of directed plane elements. They are not, however, isomorphic and therefore these bivectors are introduced as discrete and distinct elements in the notation (See Section 2.6.3, “Exterior Algebra Relationships”). This metric (+ - - -) is the Lorentz metric[18], and will be utilised throughout this thesis unless otherwise stated. The opposite of this metric, the anti-Lorentz metric (- + + +) will be used as a basis for comparison. The Lorentz (+ - - -) and the anti-Lorentz metric (- + + +) both form Minkowski spaces[41] ( $M^4$ ).

In Eq. (30), the off diagonal components no longer form the usual cross product. Here the three dimensional cross product appears as a sub-set of these components (rows 2..4, columns 2..4). i.e the outer portion of the product  $h_i h_j$ : -

$$h_i h_j = \begin{bmatrix} 1 & h_{12} & -h_{31} \\ -h_{12} & 1 & h_{23} \\ h_{31} & -h_{23} & 1 \end{bmatrix} \quad i = \{1 \dots 3\}, j = \{1 \dots 3\} \quad (31)$$

forms only part of the full four dimensional wedge product. The full outer product is referred to as the off-diagonal portion of the full four dimensional product to avoid confusion with the three dimensional case.

A four-vector within this algebra, is expressed as an Einstein summation over  $e_\mu A_\mu$ : -

$$e_\mu A_\mu = e_0 A_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} \quad (\text{Summation}) \quad (32)$$

In the above example,  $\vec{A}$  is a spatial row vector. *The reversal of row and column vector here is deliberate and implies an inner product.* The reasons for this are explained below. The basis elements are extracted out and a column vector notation is used to keep the terms compact as well as to keep track of the familiar relationships to three-space forms without the need of projecting them out. A distinction is made between terms which naturally have three components, such as  $\vec{A}$  and the 3-vector parts of 4-vectors with a vector arrow. For example the 3-vector part of the four vector  $v$  is denoted  $\vec{v}$ . Each of these corresponds to the column vector preceding it. The reverse order is used to emphasise that this should not be seen as a matrix representation\*. The advantage of the 3-component column vector notation is that it makes explicit the “3-space plus 1-time” structure of (the 4-dimensional generators of) the algebra, including their reflection properties (time-reversal and parity). Also it allows for a seamless transformation to the familiar Heaviside-Gibbs vector algebra notation.

Keeping the unit elements explicit allows for distinction of the grade of a multivector component. It should be noted that the dot and wedge products between higher grade multivectors and different grade multivectors are more complicated, but the following form always applies: -

$$FG = F \cdot G + F \wedge G \quad (33)$$

which is the sum of the symmetric ( $F \cdot G$ ) and the anti-symmetric ( $F \wedge G$ ) portions of the Clifford product.

The product of a four-vector with itself when utilising the Lorentz metric yields a different result compared to the anti-Lorentz metric. In the case of the anti-Lorentz metric the full product  $e_\mu A_\mu e_\mu A_\mu$  is: -

$$e_\mu A_\mu e_\mu A_\mu = (e_\mu A_\mu)^2 = -A_0^2 + A_1^2 + A_2^2 + A_3^2 \quad (\text{Summation}) \quad (34)$$

and for the Lorentz metric (+ - - -) the full product  $e_\mu A_\mu e_\mu A_\mu$  is: -

$$e_\mu A_\mu e_\mu A_\mu = (e_\mu A_\mu)^2 = A_0^2 - A_1^2 - A_2^2 - A_3^2 \quad (\text{Summation}) \quad (35)$$

---

\*The notation used in this thesis was developed by M.B. van der Mark. Although it has some drawbacks it has several advantages when switching to Einstein or Heaviside-Gibbs notation

Eq. (35) & (34) differ by a sign. The signs of the space terms for the Lorentz metric in Eq. (35) are negative, whereas they are positive for the anti-Lorentz case in Eq. (34). In the case of orthonormal spaces the computation of the relativistic interval  $\Delta S$ [30] (See Section 5.5, “Lorentz Invariance” for a treatment of the relativistic interval) can be calculated correctly for both metrics, without the introduction of a covariant-contravariant vector pair  $(A_\mu A^\mu)$ . Although the possibility for the introduction of co-contravariant notation is left open at this point, it is not required for the calculation of invariant lengths and intervals in this algebra. The function of the co-contravariant notation for orthogonal spaces is taken over by including the basis elements explicitly.

The motivation for utilising the Lorentz metric rather than the anti-Lorentz metric is twofold. Firstly, in the case of the anti-Lorentz metric discrepancies in the handedness of rotations have been shown unless one introduces the complex number  $i$ [18]. Additionally, time-like intervals  $(\Delta S)$  in the Lorentz metric square to give a positive scalar; the opposite sign is found for the case of the anti-Lorentz metric  $(- + + +)$ . Therefore the square root of this quantity, the invariant interval  $\sqrt{[(\Delta S)^2]}$ , is non-real for physical cases in the anti-Lorentz metric, whereas it is real in the Lorentz metric[8].

### 2.6.3 Exterior Algebra Relationships

Before proceeding any further it may be useful to consider the relationships between this Clifford algebra and some Exterior algebras such as the Grassmann algebra or the “The Algebra of Forms”[4, 37, 36, 49]. The relationship with exterior differential calculus will be given in Section 2.6.7, “Differentials”. This section will focus on the differences between the aforementioned algebras and the Clifford algebra  $\mathcal{Cl}_{1,3}$ .

Earlier in this chapter the basis elements, metric tensor and geometric product of this Clifford space were defined. These will be used here to highlight the differences between the Spacetime algebra and the Grassmann & Exterior algebras.

The primary difference between these algebras is the definition of multiplication within the algebra. The Clifford algebras  $\mathcal{Cl}_{1,3}$  and  $\mathcal{Cl}_{3,0}$  both define: -

$$e_i \cdot e_i \neq 0 \quad (\text{Clifford}) \quad (36)$$

whereas the Exterior algebras define: -

$$e_i \cdot e_i = 0 \quad (\text{Exterior}) \quad (37)$$

As a result only half the terms appear under multiplication unless a duality principle is applied. As a further consequence the diagonal of the metric tensor  $g_{\mu\nu}$  is zero everywhere for Exterior algebras. These exceptions aside, the elements of each algebra behave in the same way for the exterior product. As a consequence of these differences in the definition of multiplication in the algebra, no isomorphism can be found between the two groups.

There also exists a highly degenerate case of the class of Clifford algebras,  $\mathcal{Cl}_{0,0}$  in which the metric and inner products are zero by definition. The resulting algebra is a Grassmann algebra which is an Exterior algebra[32].

A further distinction is that the generator set of  $\mathcal{Cl}_{1,3}$ ,  $\{e_\mu\}$  does not require the introduction of a separate scalar. This element may be generated through the inner product of any element with itself. Since the square of every element is a scalar,  $\mathcal{Cl}_{1,3}$  necessarily contains all the scalars[32]. This is absent in the Exterior algebras. It is, perhaps, worth noting that introducing an interior product to an Exterior algebra may result in a Clifford algebra.

As a consequence of the fact that the behaviour of the elements differs between the Clifford and Exterior algebras, it may be misleading to make a direct correspondence between the forms and the basis elements, except in their final interpretation. This, despite the fact that they may represent the same geometric constructs and have the same geometric interpretation, a transformation from one to the other requires the redefinition of the underlying algebra, in essence moving from an Exterior to a Clifford algebra or vice versa. In summary, two elements may be used to represent the same quantities and may have a physical analogy, but no isomorphism can be found because of the differences in the algebras themselves. Therefore, to prevent confusion, a clear distinction is made between these elements in this thesis.

In addition to these relationships the Clifford algebra  $\mathcal{Cl}_{1,3}$  may be represented, for a Cartesian basis, using 2x2 matrices[18]. These matrices contain

Quaternion entries. As Hamilton's quaternions behave exactly like the bivector group  $e_{ij}$  (See Sections 2.6.5 & 2.6.9) in  $\mathcal{Cl}_{1,3}$ , the full group of 16 basis elements may be represented\* with quaternion entries[38]:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (38)$$

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \quad (39)$$

$$e_{01} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, e_{02} = \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, e_{03} = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \quad (40)$$

$$e_{23} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, e_{31} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, e_{12} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \quad (41)$$

$$e_{123} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, e_{023} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, e_{031} = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, e_{012} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} \quad (42)$$

$$e_{0123} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (43)$$

Although the basis elements may be represented by these matrices, the basis elements should not be taken to be as such in the notation throughout this thesis. If an analogy is required when thinking about these objects then it may be useful to think of the basis elements as an n-form (the bivector  $e_{0i}$  being analogous to a 2-form or the trivector  $e_{0ij}$  being analogous to a 3-form for example). As has already been discussed, however, the fact that the underlying algebras differ significantly has led to a distinction being made between the n-forms and the vectors, bivectors, trivectors etc. These elements should always be taken to be discrete elements of themselves.

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\*These matrices are a particular set of Dirac matrices, See page 256 of "Principles of Quantum Mechanics", P. M Dirac, 4th edition.

### 2.6.4 Higher Grade Objects

Starting with the set of vector basis elements  $\{e_\mu\}$ , elements of higher grade can be formed. The formation of three bivectors from the three base vectors was demonstrated earlier in this chapter. In the case of the four dimensional algebra  $(+ - - -)$ , it is possible to form 6 linearly independent terms of the form  $e_\mu \wedge e_\nu$ . These are identifiable in Eq. (30).

These bivectors may be split into two sets, those which have components only in space (space-like; denoted  $e_{ij}$ ) and those with one component in space and one component in time (which are called spacetime-like and denoted  $e_{0i}$ ). These objects have the following properties: -

$$e_{0i}^2 = e_{0i}e_{0i} = e_{0i}(-e_{i0}) = +1 \quad (44)$$

$$e_{ij}^2 = e_{ij}e_{ij} = e_{ij}(-e_{ji}) = -1 \quad (45)$$

Eq. (44) & (45) show the reduction steps required to calculate the square of  $e_{0i}$  and  $e_{ij}$ . Two adjacent sub-elements that are alike may be contracted i.e. removed and an appropriate overall sign change invoked according to the metric signature for that specific unit vector. Refer to Section 2.3, “The Clifford Product of Vectors” for an explanation of the Geometric product.

In addition, the anti-commuting properties of the algebra allow the ordering of individual sub-elements to be reversed provided the overall sign of the term is inverted. Therefore it is possible to arrange the sub-elements in such a way that all like elements are adjacent. These elements may then removed and appropriate sign changes made. For a computational and algorithmic[44, 45] treatment of this see Chapter 3, “Algebraic Reduction Algorithm”.

Trivectors and a quadrivector may also be formed in the same way as the bivectors. In this algebra there are four trivectors and a quadrivector, of which three of the trivectors contain time as a constituent component. This set of trivectors ( $e_{0ij}$ ) is referred to, in the context of this thesis, as the spacetime-like set of trivectors. The fourth trivector is the directed volume element ( $e_{123}$ ) and is similar to the pseudoscalar in three-space. However the two objects behave differently under certain transformations and, as a result, it is not appropriate to make a correspondence between the two.

The full set of basis elements can be represented as: -

$$\mathbb{1} = 1 \quad (1) \quad (46)$$

$$e_0 = e_0 \quad (1) \quad (47)$$

$$e_i = e_i \quad (1) \quad (48)$$

$$e_0 e_i = e_{0i} \quad (3) \quad (49)$$

$$e_i e_j = e_{ij} \quad (3) \quad (50)$$

$$e_0 e_i e_j = e_{0ij} \quad (3) \quad (51)$$

$$e_i e_j e_k = e_{123} \quad (1) \quad (52)$$

$$e_0 e_i e_j e_k = e_{0123} \quad (1) \quad (53)$$

When totalling the full set of linearly independent objects, there are 16 basis elements in all. A scalar, 4 vectors, 6 bivectors, 4 trivectors and a single quadrivector. This final object is a pseudoscalar in this algebra. It behaves like the scalar but differs in that it changes sign under a spatial or temporal reflection (See Section 2.10, “Conjugates” & Section 2.7, “The Duality Principle & Duality Operations”). An example of this scalar-like nature is found in the commutation behaviour; the quadrivector commutes with all basis elements which have an even number of sub-elements. Only the scalar commutes with all elements.

The multiplication of any algebraic object by any non-scalar results in a change of grade. These particular multiplication processes transform basis elements and multivectors into completely different objects. Double application of the same non-scalar object will return the original object to within a sign. This will be examined in detail in later chapters where it will be shown how one is able to set up field transformation properties utilising the algebra (See section 4.5, “The Dual Field Strength Tensor”).

### 2.6.5 Even and Odd Basis Elements

Within the Spacetime Algebra’s full group[46] of 16 basis elements (See Sections 2.4, 2.6.2 & 2.6.4) there exists a number of closed groups (See Section 2.6.9, “Groups & Subalgebras”). One such group contains only basis elements which

all have an even number of sub-indices. Proof of the closure of this group is trivial as the multiplication of any even basis element with any other will always result in an even basis element. Concentrating only on the number of sub-indices of the resultant basis element, and ignoring the trivial case of scalar multiplication, all the possible inter-multiplications of the group  $\{1, e_{\mu\nu}, e_{\mu\nu\lambda\gamma}\}$  may be represented in: -

$$e_{\mu\nu}e_{\mu\nu} = -1 \tag{54}$$

$$e_{\mu\nu}e_{\nu\lambda} = -e_{\mu\lambda} \tag{55}$$

$$e_{\mu\nu}e_{\mu\nu\lambda\gamma} = -e_{\lambda\gamma} \tag{56}$$

$$e_{\mu\nu}e_{\lambda\gamma} = e_{\mu\nu\lambda\gamma} \tag{57}$$

$$\tag{58}$$

Sign changes which occur with variations in ordering have no affect on the form of the resultant basis elements.

A closed group [46, 47] of odd basis elements under multiplication (i.e. basis elements which all have an odd number of sub-indices) cannot be defined in this algebra. The multiplication of any two non-parallel vector basis elements (odd), will yield a bivector; which is even and therefore outside the group (See Section 2.3, “The Clifford Product of Vectors”). Those basis elements which are in the full 16 basis element algebra, but which do not appear in the even group, are referred to as the odd set of basis elements.

Applications of the even group of basis elements can be found in Section 5.2, “The Lorentz Transformation” where the properties of the algebra in relativistic electromagnetism are given and generalised statements are formed regarding each elements transformation properties based on whether the element under examination is odd or even.

## 2.6.6 The Quotient

In this section, division, within the context of vectors will be defined. The meaning of the usual division notation must be explicitly defined, as division of the form  $\frac{A}{B}$  does not imply any ordering. As an example of this, take the case at



hand, “A over B”. There exists a degree of freedom in choosing whether this is separable as  $A \frac{1}{B}$  or  $\frac{1}{B} A$ , which are “A divided by B” and “B divided into A” respectively. These two are distinct and may differ by a change of sign[48]. In this thesis, “A over B” is defined as “A divided by B” unless otherwise stated. i.e.

$$\frac{A}{B} = A \frac{1}{B} \quad (59)$$

In the three-space Euclidean metric (+ + +) the quotient of a basis element ( $h_i$ ) is: -

$$h_i^2 = 1 \ ; \ h_i = 1/h_i \quad (60)$$

For the Lorentz metric (+ - - -): -

$$e_0^2 = 1 \ ; \ e_0 = 1/e_0 \quad (61)$$

and

$$e_i^2 = -1 \ ; \ e_i = -1/e_i; \ i = \{1 \dots 3\} \quad (62)$$

For the case of the anti-Lorentz metric (- + + +) these signs are reversed (See Section 2.6.2, “The Base Vector Set & Metric Tensor”).

## 2.6.7 Differentials

The differential operator in this algebra is defined: -

$$d = \lim_{\Delta x_\mu \rightarrow 0} \frac{1}{\Delta x_\mu e_\mu} \quad \mu = \{0 \dots 3\} \quad (Summation) \quad (63)$$

which can be expanded and simplified as follows: -

$$d = \frac{\partial}{\partial x_\mu e_\mu} = \partial_0 e_0 - \partial_1 e_1 - \partial_2 e_2 - \partial_3 e_3 = e_0 \partial_0 - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \nabla \quad (Summation) \quad (64)$$

Where  $\partial_\mu$  is explicitly: -

$$\partial_\mu = \lim_{\Delta x_\mu \rightarrow 0} \frac{1}{\Delta x_\mu} \quad \mu = \{0 \dots 3\} \quad (\text{Summation}) \quad (65)$$

Comparing this differential operator, with a counterpart using differential forms[49]: -

$$d_{forms} \Psi = \frac{\partial \Psi}{\partial_\mu} \mathbf{d}x^\mu \quad (\text{Summation}) \quad (66)$$

it is apparent that there is a correspondence between  $e_\mu$  and  $\mathbf{d}x^\mu$ . However, it has been demonstrated in Section 2.6.3, “Exterior Algebra Relationships” that although there is a physical correspondence between these elements, the differences in the underlying algebras means that no isomorphism exists between the differential forms and the basis elements in  $\mathcal{C}\ell_{1,3}$ .

The differential operator,  $d$  operating on an arbitrary multivector  $\Phi$  yields: -

$$d\Phi = d \cdot \Phi + d \wedge \Phi \quad (67)$$

Which is the sum of a symmetric differential and an anti-symmetric differential. For the benefit of those readers more familiar with the usual relativistic notation[30], a correspondence can be made with the differential operator operator  $d^\mu$ : -

$$d = \frac{\partial}{\partial x_\mu e_\mu} \Leftrightarrow d^\mu \quad (\text{Summation}) \quad (68)$$

As before, the column vector notation is used to keep the terms compact. The application of the differential operator  $d$  on some multivector  $\Psi$  results in a change of grade. The four-space Laplacian operator is, however, a scalar: -

$$d^2 = d \cdot d = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 = \partial_0^2 - \nabla^2 \quad (69)$$

and hence it does not change the grade of any multivector (See Section 2.6.4, “Higher Grade Objects”).

The expression  $d(dA)$  is not entirely equivalent to the second order derivative  $d^2A$ . However, in the absence of a trivector source term (which, due to the non-existence of magnetic monopoles, is always the case), the two expressions are both zero.  $dA$  is of the form: -

$$dA = d_0 A_0 + \nabla \cdot \vec{A} + \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} (d_0 \vec{A} + \nabla A_0) - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \nabla \times \vec{A} \quad (70)$$

This is easily confirmed as the second order derivative  $d^2$  is of scalar form, whereas  $d(dA)$  may contain both vector and trivector components which, as  $A$  is of vector form, requires multiplication by a non-scalar. In the case of  $d(dA)$  the two trivector terms will always cancel to give zero.

### 2.6.8 Sines, Cosines & Exponential Functions

In later chapters the spacetime algebra will be used to express differential equations which describe real physics. Although the physics will not be examined in this chapter, the form and behaviour of functions which will ultimately be used to form solutions to physical equations will be examined. Solutions of differential equations are, in many cases, combinations of sinusiodals, exponentials or other power series functions. The utility of a number of such functions will be introduced in this section and any potential problems with them highlighted. A treatment of power series functions can be found in Appendix B.

A particular basis element may be used to form a multivector sinusoidal function in the following way: -

$$\Phi_{varying} = e_\mu \sin(\theta) \quad (71)$$

More complicated expressions may be formed by simply adding in more terms:

$$\Phi_{varying} = e_\mu \sin(\theta) + e_\nu \cos(\eta) \quad (72)$$

From Appendix B, the expansion of exponential functions differs for basis elements which square to plus and minus unity. Therefore, for convenience, to show metric independence and to eliminate the need to list expressions for all 16 basis elements the two sets are expressed (See Section 2.6.4, "Higher Grade

Objects”) such that for each basis element: -

$$e_{\ominus} = \{e_{\gamma} : e_{\gamma}^2 = -1\} \quad , \quad e_{\oplus} = \{e_{\gamma} : e_{\gamma}^2 = +1\} \quad (73)$$

where  $e_{\gamma}$  in Eq. (73) is any of the 16 elements of the algebra. In Appendix B it is explicitly demonstrated that an exponential function containing members of  $e_{\ominus}$  may be expanded in the same way as a complex exponential as follows: -

$$A_0 \exp [e_{\ominus} \theta] = A_0 \cos(\theta) + A_0 e_{\ominus} \sin(\theta) \quad (74)$$

and for those containing members of  $e_{\oplus}$  the expansion is in terms of hyperbolic functions: -

$$A_0 \exp [e_{\oplus} \theta] = A_0 \cosh(\theta) + A_0 e_{\oplus} \sinh(\theta) \quad (75)$$

To prevent notational confusion “exp” with superscript indices are used throughout this thesis instead of “e” when representing exponential functions. Special care is required when working with these “hypercomplex” [8] exponentials. Due to the non-commuting properties of the algebra, the ordering of terms in the argument of an exponential function is significant when anti-commuting basis elements are introduced. Applications of these functions may be found in Section 4.6.2, “Standard Wave Functions”; Section 4.6.5, “New Wave Functions” & Section 5.2, “The Lorentz Transformation”.

### 2.6.9 Groups & Subalgebras

A group may defined and created with a group generator. A set of generators  $\{g_0 \dots g_n\}$  is a set of group elements such that repeated application of the generators on themselves and each other are capable of producing all the elements in the group.

The means by which the sixteen basis elements of the spacetime algebra are formed has been given in Section 2.6.4, “Higher Grade Objects”. These basis elements (scalar, vector, bivector, trivector and quadrivector) are formed from the minimal generator set  $\{e_{\mu}\}$ . This generator forms the full group under multiplication.

Within the full group there are a number of sub-groups which are themselves

closed. A group is said to be closed under multiplication if and only if there is no possible multiplication, or combinations of multiplications that yields an object which is outside the group.

Isomorphism	Generator	Objects in Group
Spacetime Algebra	$\{e_\mu\}$	16, $\{1, e_0, e_i, e_{0i}, e_{ij}, e_{0ij}, e_{123}, e_{0123}\}$
Even Subalgebra	$\{e_{0i}\}$	8, $\{1, e_{0i}, e_{ij}, e_{0123}\}$
“Quaternion Algebra”	$\{e_{ij}\}$	4, $\{1, e_{ij}\}$
“Complex Numbers”	$\{e_{0123}\}$	2, $\{1, e_{0123}\}$
Real Numbers	$\{1\}$	1, $\{1\}$

**Table 2.2.** Examples of closed groups within the real Dirac algebra with which an isomorphism can be found to particular algebras.

Table 2.2 gives some examples of groups and the sub-algebras they may be used to represent. There are many other groups which are closed, some of which are examined in Section 2.9, “Properties of the Spacetime Algebra”. The investigation of various possible groups within the Spacetime algebra will not be pursued further here.

### 2.6.10 Discussion & Summary

The basic properties of the algebra which will be utilised in this thesis have been introduced in this section. It has been shown that by choosing the Lorentz metric (+ - - -), the introduction of a covariant-contravariant multiplication system is not required in order to calculate the relativistic interval (See Section 5.5, “Lorentz Invariance” for a fuller discussion of the interval). In addition, how the four unit vectors ( $e_\mu$ ) may combine, under multiplication, to give the 16 linearly independent basis elements, including those in the form of directed plane and volume elements has been demonstrated.

The division of simple basis elements has been addressed in this section. The treatment of division presented has revealed that, unless properly defined, the expression  $\frac{A}{B}$  is ambiguous in non-commuting algebras. Clarification is achieved by rigorously defining “A over B” as “A divided by B” i.e  $\frac{A}{B} = A \frac{1}{B}$  unless otherwise stated. Sinusoidal and exponential functions have also been introduced and

a discussion on the care required in dealing with such functions with the algebra presented here has been given. Groups and power series functions have also been examined.

Finally the differential operator has been introduced and a demonstration given that the first order differentiation of a quantity incurs a change in grade of an object, i.e. the four differential of a scalar results in a four-vector. Similarly, the fact that the second order differential operator  $d^2$  has no effect on the grade of an object has also been illustrated. For a treatment of higher order differentials see Section 5.3, "Proper Quantities & the Covariant Derivative".

## 2.7 The Duality Principle & Duality Operations

### 2.7.1 Introduction

In this section the duality principle[50, 51] will be introduced and the concept of duality operations[27, 30, 41] will be discussed. The duality operation which will be utilised here is an extended implentational form of the *duality principle*. The duality principle states that all the propositions in projective geometry occur in dual pairs which have the property that, starting from either proposition of a pair, the other can be immediately inferred by interchanging the parts played by the words "point" and "line"[50, 51] for two dimensional geometry; or point and plane for three dimensional geometry. The duality principle was first discussed by Joseph Gergonne[50] in 1826.

In higher dimensional spaces, the duality operation maps objects of form N onto object of form M, where N and M may or may not be the same. A geometric proposition is said to be self-dual when application of the duality principle of projective geometry results in a proposition equivalent to the original.

The Hodge duality operator[30, 27, 41, 52] maps an object with form of degree k to an object with form of degree n-k, where n is the dimension of the space. The double application of this operator must lead back to the original form up to within a factor. The sign of this factor may be dependent on the choice of metric. Therefore the Hodge duality operation yields different results when utilised in different algebras and projections.

In this section the behaviour of the Hodge duality operator in three and four

dimensions will be examined. This will be conducted by utilising the three-space Euclidean metric to begin with before progressing onto the four-space Lorentzian (+ - - -) case.

## 2.7.2 Dual in Three-Space

The behaviour of the Hodge dual[30, 27, 41, 35, 8] in terms of the three-space Euclidean algebra ( $h_\mu$ ) will be demonstrated. The Hodge dual is an operation which maps an object of form  $k$  onto an object of form  $n-k$ . In three dimensional space, the Hodge dual of the vector  $h_i$  is a 2-form object i.e. a bivector. That this is specifically the bivector which spans the remainder of the space, excluding the original vector, has been demonstrated by Lounesto[27]. In the three-space Clifford algebra, the duality operator can be implemented by the multiplicative application of the basis element which spans the whole space, in this case the  $h_{123}$  element. Therefore the Hodge dual of the element  $h_i$  may be calculated as: -

$$Dual(h_i) = h_i h_{123} = h_{123} h_i = h_{jk} \quad (76)$$

In this specific example the ordering of the implementation of the duality operation makes no difference to the overall result. An graphical example of the Hodge dual operating on a vector in three dimensional space can be seen in Figure. 2.1. This operation is akin to a transformation to and/or from a vector and a bivector. Similarly, the dual of the bivector  $h_{jk}$  may be defined as: -

$$Dual(h_{jk}) = h_{jk} h_{123} = -h_i \quad (77)$$

demonstrating that the double application of the dual operator leads back to the original form within a factor. The fact that there is a change of sign in the reverse operation is a consequence of the ordering. Hence: -

$$Dual(Dual(h_i)) = Dual(h_{jk}) = h_i h_{123} h_{123} = -h_i \quad (78)$$

Eq. (78) shows the origin of the reversal in sign with greater clarity. Note that in this three dimensional Euclidean space, the square of the dual is minus one ( $h_{123}^2 = -1$ ). The final possible object which has a dual, in this space, is the scalar itself. The dual of the scalar is, by definition, that object which spans the

complete space, in this case  $h_{123}$ . Again, the double application of this duality operator leads to the original form within a factor, which is again a change in the overall sign.

### 2.7.3 Dual in Four-Space

In this section the duality operator in the four dimensional Lorentzian space will be examined. The duality operator is implemented in this instance using the object which spans the whole space. In the case of four-space in this case it is the quadrivector  $e_{0123}$ .

Element	→	Dual (Pre)	Dual (Post)
1	→	$e_{0123}$	$e_{0123}$
$e_0$	→	$-e_{123}$	$e_{123}$
$e_i$	→	$-e_{0jk}$	$e_{0jk}$
$e_{0i}$	→	$e_{jk}$	$e_{jk}$
$e_{jk}$	→	$-e_{0i}$	$-e_{0i}$
$e_{0jk}$	→	$e_i$	$-e_i$
$e_{123}$	→	$e_0$	$-e_0$
$e_{0123}$	→	-1	-1

**Table 2.3.** This table shows the pre-multiply and post-multiply duality transformations. Notice that the forward and reverse transformations differ in each case by a change of sign. Also notice that the sign for the two duality transformations is the same for the even basis elements, but differs by a sign for the odd basis elements.

Table 2.3 lists the transformation properties of the 16 basis elements under the duality transforms. Notice that the pre- or post-multiplicative application of the  $e_{0123}$  object affects the sign of certain results and not others. Pre or Post application of the duality object is defined explicitly as : -

$$Dual(\Psi) = e_{0123} \Psi \text{ (pre)} \quad (79)$$

$$Dual(\Psi) = \Psi e_{0123} \text{ (post)} \quad (80)$$

For all even components of an arbitrary multivector  $\Psi$  (See Section 2.6.5), the ordering (pre or post) has no effect on the result because the quadrivector



$e_{0123}$  (as a whole) commutes with all even elements in much the same way as a scalar. However, the pseudoscalar anti-commutes with all odd basis elements ( $e_0$ ,  $e_i$ ,  $e_{0ij}$  &  $e_{123}$ ) and this therefore necessitates the introduction of the pre and post multiply dual operations. Note that no matter whether one chooses the duality operator as a post or pre multiplicative system, the double application of the operator always yields the original object with a sign change. Therefore there is always a change in sign invoked by the application of this duality operator. Note that in the four dimensional case the  $e_{123}$  is not a pseudoscalar and the commutation properties are different to the three dimensional case.

A choice must always be made with regard to the duality operator at this point, regardless of the algebra in question. In this thesis the Hodge dual is identified with that operator[38] which for a multivector  $\Psi$ , transforms the object as: -

$$\bar{\Psi} = -e_{0123}\Psi \quad (81)$$

The factor of -1 is introduced in Eq. (81) to make a correspondence with the Hodge dual in relativistic notation[30](See Section 4.5, “The Dual Field Strength Tensor”).

## 2.7.4 Discussion & Summary

In this section the duality principle[50] has been introduced and utilised, along with the definition of the Hodge dual[27, 30, 41] to explain the concept of the duality operator. Here the Hodge dual has been introduced firstly in the three dimensional Euclidean metric before progressing on to a description of the dual in the four dimensional Lorentz metric. How the various basis elements are the dual of others (in particular algebras) and how the double application of the duality operator leads back to the same form object (to within a factor) has been shown. Further discussions of duality operations can be seen in Section 4.5, “The Dual Field Strength Tensor”.

## 2.8 Parallel Objects

The concepts of “parallelness” and perpendicularity of the basis elements amongst the sixteen basis elements of this algebra are linearly independent, there is a historical case for identifying whether quantities represented by these elements are “parallel” or “perpendicular”. There are historical reasons for identifying cases where  $\mathbf{e}_i \parallel \mathbf{e}_j$  and  $\mathbf{e}_i \perp \mathbf{e}_j$ . As  $\mathbf{e}_i$  and  $\mathbf{e}_j$  are composed of the sets  $\{ \mathbf{e}_0i \}$  and  $\{ \mathbf{e}_ij \}$  respectively (See Section 4.2, “Potentials & Fields”) they are always linearly independent of each another, regardless of their direction, in this algebra. Nevertheless there is a case for identifying where  $\mathbf{e}_i$  and  $\mathbf{e}_j$  are “parallel” or “perpendicular” in terms of their three-space visualisations. This is accomplished by projecting these elements onto three-space. The direction of  $\mathbf{e}_i$  (A space-like bivector) is a choice. Here the historical choice of projecting  $\mathbf{e}_i$  onto a “turning vector” enables a comparison to be drawn. Note that “parallelness” does not imply that the elements are truly parallel, it simply infers that the elements are similarly directed when projected into their “historical” three-space form. These projections are only used for making comparisons.

The motivation for examining these attributes is found in their utility in the investigation of symmetries of relativistic transformations (See Section 5.2, “The Lorentz Transformation”).

The aim of this section is to present a sensible and physical argument for a projection of each of the 16 basis elements, where possible, onto one of the  $\hat{x}$ ,  $\hat{y}$  or  $\hat{z}$  axes. A mapping of the basis elements onto a three dimensional system will then be used to extrapolate parallel and perpendicular attributes. The arguments will be presented in terms of properties of elements when the duality principle and the Hodge dual from Section 2.7 are applied to them. Similarly, the spacetime

$$dA = d_0 A_0 + \nabla \cdot \vec{A} + \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} (d_0 \vec{A} + \nabla A_0) - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \nabla \times \vec{A} \quad (70)$$

This is easily confirmed as the second order derivative  $d^2$  is of scalar form, whereas  $d(dA)$  may contain both vector and trivector components which, as  $A$  is of vector form, requires multiplication by a non-scalar. In the case of  $d(dA)$  the two trivector terms will always cancel to give zero.

### 2.6.8 Sines, Cosines & Exponential Functions

In later chapters the spacetime algebra will be used to express differential equations which describe real physics. Although the physics will not be examined in this chapter, the form and behaviour of functions which will ultimately be used to form solutions to physical equations will be examined. Solutions of differential equations are, in many cases, combinations of sinusiodals, exponentials or other power series functions. The utility of a number of such functions will be introduced in this section and any potential problems with them highlighted. A treatment of power series functions can be found in Appendix B.

A particular basis element may be used to form a multivector sinusoidal function in the following way: -

$$\Phi_{varying} = e_\mu \sin(\theta) \quad (71)$$

More complicated expressions may be formed by simply adding in more terms:

$$\Phi_{varying} = e_\mu \sin(\theta) + e_\nu \cos(\eta) \quad (72)$$

From Appendix B, the expansion of exponential functions differs for basis elements which square to plus and minus unity. Therefore, for convenience, to show metric independence and to eliminate the need to list expressions for all 16 basis elements the two sets are expressed (See Section 2.6.4, "Higher Grade

Objects”) such that for each basis element: -

$$e_{\ominus} = \{e_{\gamma} : e_{\gamma}^2 = -1\} \quad , \quad e_{\oplus} = \{e_{\gamma} : e_{\gamma}^2 = +1\} \quad (73)$$

where  $e_{\gamma}$  in Eq. (73) is any of the 16 elements of the algebra. In Appendix B it is explicitly demonstrated that an exponential function containing members of  $e_{\ominus}$  may be expanded in the same way as a complex exponential as follows: -

$$A_0 \exp[e_{\ominus} \theta] = A_0 \cos(\theta) + A_0 e_{\ominus} \sin(\theta) \quad (74)$$

and for those containing members of  $e_{\oplus}$  the expansion is in terms of hyperbolic functions: -

$$A_0 \exp[e_{\oplus} \theta] = A_0 \cosh(\theta) + A_0 e_{\oplus} \sinh(\theta) \quad (75)$$

To prevent notational confusion “exp” with superscript indices are used throughout this thesis instead of “e” when representing exponential functions. Special care is required when working with these “hypercomplex” [8] exponentials. Due to the non-commuting properties of the algebra, the ordering of terms in the argument of an exponential function is significant when anti-commuting basis elements are introduced. Applications of these functions may be found in Section 4.6.2, “Standard Wave Functions”; Section 4.6.5, “New Wave Functions” & Section 5.2, “The Lorentz Transformation”.

## 2.6.9 Groups & Subalgebras

A group may be defined and created with a group generator. A set of generators  $\{g_0 \dots g_n\}$  is a set of group elements such that repeated application of the generators on themselves and each other are capable of producing all the elements in the group.

The means by which the sixteen basis elements of the spacetime algebra are formed has been given in Section 2.6.4, “Higher Grade Objects”. These basis elements (scalar, vector, bivector, trivector and quadrivector) are formed from the minimal generator set  $\{e_{\mu}\}$ . This generator forms the full group under multiplication.

Within the full group there are a number of sub-groups which are themselves

closed. A group is said to be closed under multiplication if and only if there is no possible multiplication, or combinations of multiplications that yields an object which is outside the group.

Isomorphism	Generator	Objects in Group
Spacetime Algebra	$\{e_\mu\}$	16, $\{1, e_0, e_i, e_{0i}, e_{ij}, e_{0ij}, e_{123}, e_{0123}\}$
Even Subalgebra	$\{e_{0i}\}$	8, $\{1, e_{0i}, e_{ij}, e_{0123}\}$
“Quaternion Algebra”	$\{e_{ij}\}$	4, $\{1, e_{ij}\}$
“Complex Numbers”	$\{e_{0123}\}$	2, $\{1, e_{0123}\}$
Real Numbers	$\{1\}$	1, $\{1\}$

**Table 2.2.** Examples of closed groups within the real Dirac algebra with which an isomorphism can be found to particular algebras.

Table 2.2 gives some examples of groups and the sub-algebras they may be used to represent. There are many other groups which are closed, some of which are examined in Section 2.9, “Properties of the Spacetime Algebra”. The investigation of various possible groups within the Spacetime algebra will not be pursued further here.

### 2.6.10 Discussion & Summary

The basic properties of the algebra which will be utilised in this thesis have been introduced in this section. It has been shown that by choosing the Lorentz metric (+ - - -), the introduction of a covariant-contravariant multiplication system is not required in order to calculate the relativistic interval (See Section 5.5, “Lorentz Invariance” for a fuller discussion of the interval). In addition, how the four unit vectors ( $e_\mu$ ) may combine, under multiplication, to give the 16 linearly independent basis elements, including those in the form of directed plane and volume elements has been demonstrated.

The division of simple basis elements has been addressed in this section. The treatment of division presented has revealed that, unless properly defined, the expression  $\frac{A}{B}$  is ambiguous in non-commuting algebras. Clarification is achieved by rigorously defining “A over B” as “A divided by B” i.e  $\frac{A}{B} = A \frac{1}{B}$  unless otherwise stated. Sinusoidal and exponential functions have also been introduced and

a discussion on the care required in dealing with such functions with the algebra presented here has been given. Groups and power series functions have also been examined.

Finally the differential operator has been introduced and a demonstration given that the first order differentiation of a quantity incurs a change in grade of an object, i.e. the four differential of a scalar results in a four-vector. Similarly, the fact that the second order differential operator  $d^2$  has no effect on the grade of an object has also been illustrated. For a treatment of higher order differentials see Section 5.3, "Proper Quantities & the Covariant Derivative".

## 2.7 The Duality Principle & Duality Operations

### 2.7.1 Introduction

In this section the duality principle[50, 51] will be introduced and the concept of duality operations[27, 30, 41] will be discussed. The duality operation which will be utilised here is an extended implementational form of the *duality principle*. The duality principle states that all the propositions in projective geometry occur in dual pairs which have the property that, starting from either proposition of a pair, the other can be immediately inferred by interchanging the parts played by the words "point" and "line"[50, 51] for two dimensional geometry; or point and plane for three dimensional geometry. The duality principle was first discussed by Joseph Gergonne[50] in 1826.

In higher dimensional spaces, the duality operation maps objects of form  $N$  onto object of form  $M$ , where  $N$  and  $M$  may or may not be the same. A geometric proposition is said to be self-dual when application of the duality principle of projective geometry results in a proposition equivalent to the original.

The Hodge duality operator[30, 27, 41, 52] maps an object with form of degree  $k$  to an object with form of degree  $n-k$ , where  $n$  is the dimension of the space. The double application of this operator must lead back to the original form up to within a factor. The sign of this factor may be dependent on the choice of metric. Therefore the Hodge duality operation yields different results when utilised in different algebras and projections.

In this section the behaviour of the Hodge duality operator in three and four

dimensions will be examined. This will be conducted by utilising the three-space Euclidean metric to begin with before progressing onto the four-space Lorentzian (+ - - -) case.

### 2.7.2 Dual in Three-Space

The behaviour of the Hodge dual[30, 27, 41, 35, 8] in terms of the three-space Euclidean algebra ( $h_\mu$ ) will be demonstrated. The Hodge dual is an operation which maps an object of form  $k$  onto an object of form  $n-k$ . In three dimensional space, the Hodge dual of the vector  $h_i$  is a 2-form object i.e. a bivector. That this is specifically the bivector which spans the remainder of the space, excluding the original vector, has been demonstrated by Lounesto[27]. In the three-space Clifford algebra, the duality operator can be implemented by the multiplicative application of the basis element which spans the whole space, in this case the  $h_{123}$  element. Therefore the Hodge dual of the element  $h_i$  may be calculated as: -

$$Dual(h_i) = h_i h_{123} = h_{123} h_i = h_{jk} \quad (76)$$

In this specific example the ordering of the implementation of the duality operation makes no difference to the overall result. An graphical example of the Hodge dual operating on a vector in three dimensional space can be seen in Figure. 2.1. This operation is akin to a transformation to and/or from a vector and a bivector. Similarly, the dual of the bivector  $h_{jk}$  may be defined as: -

$$Dual(h_{jk}) = h_{jk} h_{123} = -h_i \quad (77)$$

demonstrating that the double application of the dual operator leads back to the original form within a factor. The fact that there is a change of sign in the reverse operation is a consequence of the ordering. Hence: -

$$Dual(Dual(h_i)) = Dual(h_{jk}) = h_i h_{123} h_{123} = -h_i \quad (78)$$

Eq. (78) shows the origin of the reversal in sign with greater clarity. Note that in this three dimensional Euclidean space, the square of the dual is minus one ( $h_{123}^2 = -1$ ). The final possible object which has a dual, in this space, is the scalar itself. The dual of the scalar is, by definition, that object which spans the

complete space, in this case  $h_{123}$ . Again, the double application of this duality operator leads to the original form within a factor, which is again a change in the overall sign.

### 2.7.3 Dual in Four-Space

In this section the duality operator in the four dimensional Lorentzian space will be examined. The duality operator is implemented in this instance using the object which spans the whole space. In the case of four-space in this case it is the quadrivector  $e_{0123}$ .

Element	$\rightarrow$	Dual (Pre)	Dual (Post)
1	$\rightarrow$	$e_{0123}$	$e_{0123}$
$e_0$	$\rightarrow$	$-e_{123}$	$e_{123}$
$e_i$	$\rightarrow$	$-e_{0jk}$	$e_{0jk}$
$e_{0i}$	$\rightarrow$	$e_{jk}$	$e_{jk}$
$e_{jk}$	$\rightarrow$	$-e_{0i}$	$-e_{0i}$
$e_{0jk}$	$\rightarrow$	$e_i$	$-e_i$
$e_{123}$	$\rightarrow$	$e_0$	$-e_0$
$e_{0123}$	$\rightarrow$	-1	-1

**Table 2.3.** This table shows the pre-multiply and post-multiply duality transformations. Notice that the forward and reverse transformations differ in each case by a change of sign. Also notice that the sign for the two duality transformations is the same for the even basis elements, but differs by a sign for the odd basis elements.

Table 2.3 lists the transformation properties of the 16 basis elements under the duality transforms. Notice that the pre- or post-multiplicative application of the  $e_{0123}$  object affects the sign of certain results and not others. Pre or Post application of the duality object is defined explicitly as : -

$$Dual(\Psi) = e_{0123} \Psi \quad (pre) \quad (79)$$

$$Dual(\Psi) = \Psi e_{0123} \quad (post) \quad (80)$$

For all even components of an arbitrary multivector  $\Psi$  (See Section 2.6.5), the ordering (pre or post) has no effect on the result because the quadrivector



$e_{0123}$  (as a whole) commutes with all even elements in much the same way as a scalar. However, the pseudoscalar anti-commutes with all odd basis elements ( $e_0$ ,  $e_i$ ,  $e_{0ij}$  &  $e_{123}$ ) and this therefore necessitates the introduction of the pre and post multiply dual operations. Note that no matter whether one chooses the duality operator as a post or pre multiplicative system, the double application of the operator always yields the original object with a sign change. Therefore there is always a change in sign invoked by the application of this duality operator. Note that in the four dimensional case the  $e_{123}$  is not a pseudoscalar and the commutation properties are different to the three dimensional case.

A choice must always be made with regard to the duality operator at this point, regardless of the algebra in question. In this thesis the Hodge dual is identified with that operator[38] which for a multivector  $\Psi$ , transforms the object as: -

$$\bar{\Psi} = -e_{0123}\Psi \quad (81)$$

The factor of -1 is introduced in Eq. (81) to make a correspondence with the Hodge dual in relativistic notation[30](See Section 4.5, “The Dual Field Strength Tensor”).

## 2.7.4 Discussion & Summary

In this section the duality principle[50] has been introduced and utilised, along with the definition of the Hodge dual[27, 30, 41] to explain the concept of the duality operator. Here the Hodge dual has been introduced firstly in the three dimensional Euclidean metric before progressing on to a description of the dual in the four dimensional Lorentz metric. How the various basis elements are the dual of others (in particular algebras) and how the double application of the duality operator leads back to the same form object (to within a factor) has been shown. Further discussions of duality operations can be seen in Section 4.5, “The Dual Field Strength Tensor”.

## 2.8 Parallel Objects

The concepts of “parallelness” and perpendicularity of the basis elements amongst themselves will be examined and defined in this section. Although the sixteen basis elements of this algebra are linearly independent, there is a historical case for identifying whether quantities represented by these elements are “parallel” or “perpendicular”. An example of this is the fields  $\vec{E}$  and  $\vec{B}$ . There are historical reasons for identifying cases where  $\vec{E} \parallel \vec{B}$  and  $\vec{E} \perp \vec{B}$ . As  $\vec{E}$  and  $\vec{B}$  are composed of the sets  $\{e_{0i}\}$  and  $\{e_{ij}\}$  respectively (See Section 4.2, “Potentials & Fields”) they are always linearly independent of each another, regardless of their direction, in this algebra. Nevertheless there is a case for identifying where  $\vec{E}$  and  $\vec{B}$  are “parallel” or “perpendicular” in terms of their three-space visualisations. This is accomplished by projecting these elements onto three-space. The direction of  $\vec{B}$  (A space-like bivector) is a choice. Here the historical choice of projecting  $\vec{B}$  onto a “turning vector” enables a comparison to be drawn. Note that “parallelness” does not imply that the elements are truly parallel, it simply infers that the elements are similarly directed when projected into their “historical” three-space form. These projections are only used for making comparisons.

The motivation for examining is these attributes is found in their utility in the investigation of symmetries of relativistic transformations (See Section 5.2, “The Lorentz Transformation”).

The aim of this section is to present a sensible and physical argument for a projection of each of the 16 basis elements, where possible, onto one of the  $\hat{x}$ ,  $\hat{y}$  or  $\hat{z}$  axes. A mapping of the basis elements onto a three dimensional system will then be used to extrapolate parallel and perpendicular attributes. The arguments will be presented in terms of properties of elements when the duality principle and the Hodge dual from Section 2.7 are applied to them. Similarly, the space-time split[34] will be utilised for a particular set of projections. There will be discussion where no sensible projection onto three-space is apparent.

The quadrivector  $e_{0123}$  is projected onto the scalar as the quadrivector’s dual is a scalar. For similar reasons the unit space volume element  $e_{123}$  is projected onto  $e_0$ , both of which, for the sake of obtaining a three dimensional system, are projected to the scalar. For all vector and spacetime bivectors, a one to one mapping can be defined between those basis elements containing the Latin

subscripts  $i, j, k$  and the three dimensional unit vectors  $\hat{x}, \hat{y}, \hat{z}$  by projecting out time. This can be seen in Eq. (82), (83) & (84).

$$e_i \rightarrow \hat{x}, \quad e_{0i} \rightarrow \hat{x} \tag{82}$$

$$e_j \rightarrow \hat{y}, \quad e_{0j} \rightarrow \hat{y} \tag{83}$$

$$e_k \rightarrow \hat{z}, \quad e_{0k} \rightarrow \hat{z} \tag{84}$$

Note that the mapping in Eq. (82), (83) & (84) is a projection and this projected set is not isomorphic to the original elements. A mapping can be found for the spacelike bivectors, by performing a duality transformation on each element. In doing so a suitable spacetime bivector for the mapping is discovered:-

$$e_{ij} \rightarrow e_{0k} \tag{85}$$

$$e_{ki} \rightarrow e_{0j} \tag{86}$$

$$e_{jk} \rightarrow e_{0i} \tag{87}$$

In Eq. (85), (86) & (87) a projection removing time (i.e. a space-time split[34]) yields a unit vector in three-space as shown. Therefore a projective mapping of twelve elements onto unit vectors in three-space has been made. The remaining four elements have been projected onto the scalar and no attempt at interpretation will be made from this. Within this scope the elements with common mappings are referred to as parallel, otherwise they are referred to as being perpendicular as shown in Table. 2.4 below.

			→		
	$\hat{i}$	$e_i$	$e_{0i}$	$e_{jk}$	$e_{0jk}$
⊥ ↓	$\hat{x}$	$e_1$	$e_{01}$	$e_{23}$	$e_{023}$
	$\hat{y}$	$e_2$	$e_{02}$	$e_{31}$	$e_{031}$
	$\hat{z}$	$e_3$	$e_{03}$	$e_{12}$	$e_{012}$

**Table 2.4.** This table shows the perpendicularity/"parallelness" of basis elements. Elements contained within a horizontal row are parallel, otherwise they are perpendicular. For elements not in this table refer to the text.

In Table 2.4 any two basis elements which lie on the same row are considered to have a common mapping and are therefore considered parallel. An expression of greater precision might be to state that they have equivalent projections onto an  $n$ -form object, at least to within a sign. In addition, should two basis elements lie on different rows in Table 2.4, then the objects would be perpendicular if projected onto a  $n$ -form object, and are therefore considered to be perpendicular to one another.

The perpendicularity or “parallelness” of any basis element outwith this table ( $1, e_0, e_{123}, e_{0123}$ ) with respect to those within the table is undefined. One may consider the scalar and pseudoscalar to be parallel. The basis elements  $e_0$  and  $e_{123}$  both project into the  $\hat{t}$  direction, and hence may be considered to be parallel to each other. A further justification of these assignments is that they have the same properties under Lorentz transformation (See Section 5.2.2, “Lorentz Boosts”).

By utilising the duality principle and the duality operator from Section 2.7, a mapping of twelve of the basis elements on to a right handed orthonormal set  $\{\hat{x}, \hat{y}, \hat{z}\}$  has been generated. This mapping has been used to assign relative “parallelness” and perpendicularity attributes between various basis elements based on whether or not any two objects map to the same component of  $\{\hat{x}, \hat{y}, \hat{z}\}$ . This has led to the production of a table which can be used as reference for such attributes. Applications of this table can be found in Chapter 5.

## 2.9 Properties of the Spacetime Algebra

### 2.9.1 Introduction

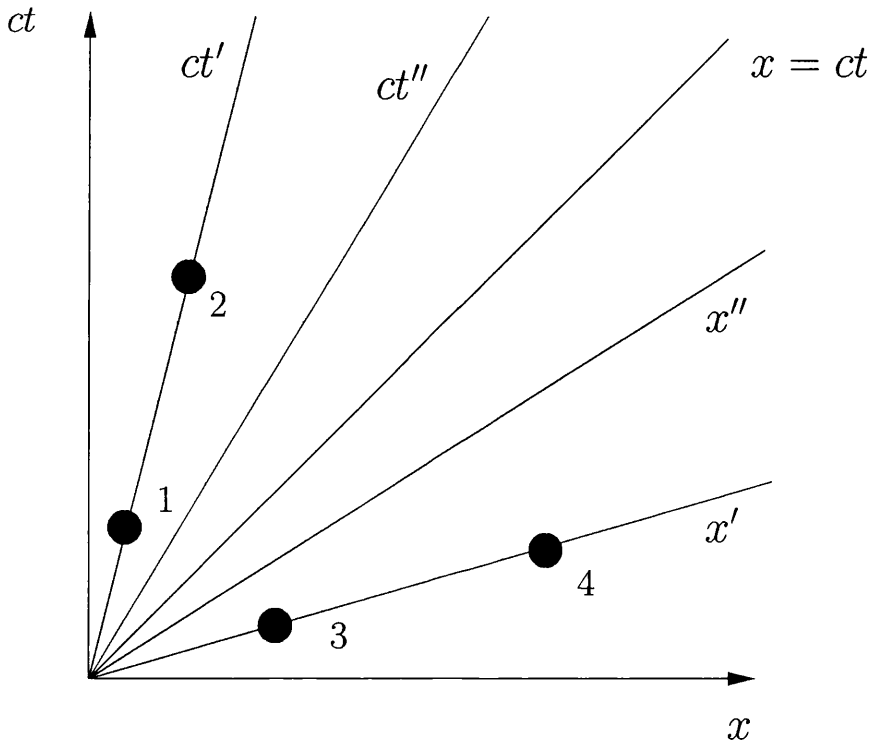
The particular properties of the algebra presented in Section 2.6 will be examined in this section in an effort to ascertain how this algebra behaves when compared to others. The defining characteristic of the algebras which bear the names of Minkowski, Lorentz and Dirac will be introduced before attempting to ascertain whether and to what degree the spacetime algebra reflects these. The Quaternion algebra will be introduced and identified as a subalgebra of the spacetime algebra.

Furthermore, as an extension to section 2.6.9, “Groups & Subalgebras”, the properties of the Lorentz and Poincaré groups will be investigated in the context of the algebra presented here. The Lorentz group will also be utilised in the

development of the Lorentz transformation in Chapter 5.

## 2.9.2 Minkowski Spacetime

A Minkowski space is a four-dimensional real vector space with metric signature  $(+ - - -)$ . The overall sign is taken to be a matter of convention and the opposite signature  $(- + + +)$  is also a Minkowski space. The Minkowski space is often denoted  $\mathbb{R}^{1,3}$  to emphasise the signature, although it is more usual to denote a Minkowski space as  $\mathbb{M}^4$ [53]. Eq. (30) shows clearly a  $(+ - - -)$  metric signature and therefore the algebra presented in this thesis is a Minkowski space.



**Figure 2.5.** Lineland Minkowski diagram showing time-like (1 & 2) and spacelike (3 & 4) events. The  $x = ct$  line denotes the “null-vector” points and represents the surface of the light cone when a flatland projection is made..

Within Minkowski spaces, vectors are classified according to the sign of their norm squared  $\Lambda = (v_0 + \vec{v})^2$ . For the  $(+ - - -)$  metric, vectors are said

to be time-like or space-like if their norms squared are positive or negative respectively[53, 39, 40]. Vectors with zero norm are called null or light-like. The set of all light-like vectors constitutes the light cone[53, 39, 40, 54]. The terminology employed here, stems from the use of Minkowski spaces in the theory of relativity[39, 40].

Figure 2.5 shows how events may be classified graphically. For any time-like pair of events (1 & 2) it is possible to find some observer ( $ct'$ ) such that the two events occur at the same location ( $x$ ) and represent a pure time interval. Hence the name time-like. Similarly, for events (3 & 4) it is possible to find an observer ( $\hat{x}'$ ) such that the events take place simultaneously and represent a pure space-like interval. The mathematical representation of such transformations are known as Lorentz transformations. For fuller discussions of Lorentz transformations refer to Chapter 5.

### 2.9.3 The Quaternion Algebra

Hamilton's quaternion algebra can be defined by the following relations : -

$$i^2 = j^2 = k^2 = ijk = -1 \quad (88)$$

A salient property of the quaternion algebra is that members of the basis element set ( $\{i, j, k\}$ ) anti-commute with each other. i.e.

$$ji = -jiijk = -k = ijkk = -ij \quad (89)$$

and

$$ij = -ijkk = k \quad (90)$$

Thus these elements (together with the scalar) form the closed quaternion group. These quaternions are isomorphic to the bivector group  $e_{ij}$  (See Sections 2.6.5 & 2.6.9) in the spacetime algebra. For convenience the Latin character denoting the quaternion is mapped to the bivector which lacks that particular character, i.e. the cyclic permutations of  $\{jk, ki, ij\}$ : -

$$i \rightarrow e_{jk} \quad (91)$$

$$j \rightarrow e_{ki} \quad (92)$$

$$k \rightarrow e_{ij} \quad (93)$$

In the case of the bivector rotor group it is clear that the group is closed as multiplication of any two objects will always yield either another rotor bivector or the scalar. Furthermore, by utilising the above examples, the fact that the mapping is isomorphic can be demonstrated. From Eq. (88): -

$$e_{jk}^2 = -e_{ik}^2 = e_{ij}^2 = e_{jk}(e_{ki})e_{ij} = -1 \quad (94)$$

and Eq. (89): -

$$e_{jk}(-e_{ik}) = e_{jkk}i = -e_{ji} = e_{ij} \quad (95)$$

where  $e_{ij}$  maps to  $k$ . This demonstrates that the rotor bivector group together with the scalar itself form a closed subalgebra which is isomorphic to the Quaternion algebra.

The quaternion algebra has been shown to be isomorphic to the group of spatial rotations  $O(3)$  by Abonyi *et al*[55]. Given that the rotor bivector group is isomorphic to the quaternion algebra, it is also isomorphic to  $O(3)$ .

## 2.9.4 Lorentz & Poincaré Groups

The Lorentz group[56] is the group of all Lorentz transformations of Minkowski spacetime[39, 40](See also section 5.2). It is the subgroup of the Poincaré group[57, 58, 53]\*  $P(1, 3)$  consisting of all isometries that leave the origin fixed[56, 59, 60], i.e. the group does not contain the translational elements. This group may be referred to as the homogeneous Lorentz group and the Poincaré group as the inhomogeneous Lorentz group[56].

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\*Note that in the strictest mathematical sense the Poincaré group is a ring rather than a group[24, 25, 46, 47]. It shall, however, be referred to it as a group as this is the usual terminology.

The Lorentz group  $O(1, 3)$  has four connected components. The elements in each component are characterised by whether or not they reverse the orientation of space and/or time. A Lorentz transformation which reverses either the orientation of time or space (but not both) has determinant  $-1$ , while the rest have determinant  $+1$ . A proper Lorentz transformation is one with determinant  $+1$ . The subgroup of proper Lorentz transformations is denoted  $SO(1, 3)$ . A transformation which preserves the orientation of time (relative to the orientation of space) is called orthochronous. The Lorentz group has at least four disconnected components which can be classified as: -

Proper Orthochronous  $e_0^0 > 1, \det = 1$ ;

Improper Orthochronous  $e_0^0 > 1, \det = -1$ ;

Proper Anti-Orthochronous  $e_0^0 < -1, \det = 1$ ;

Improper Anti-Orthochronous  $e_0^0 < -1, \det = -1$ ;

The subgroup of orthochronous transformations is often denoted  $O^+(3, 1)$ . This set of transformations corresponds to those which may be observed physically. As a result it is often called the restricted or proper, orthochronous Lorentz group[61], and is denoted by  $SO^+(1, 3)$ .

In the spacetime algebra a correspondence between the Lorentz group and the full set of six bivectors can be made. Isomorphisms between the spacetime bivector group (including the scalar) and the group of orthogonal rotations  $O(3)$  were given in section 2.9.3. Based on the fact that  $O(3)^+$  is a subgroup of the continuous Lorentz group,  $SO(1, 3)$ [55], the extension of this group to a closed group isomorphic to the full Lorentz group requires the introduction only of the  $e_{0i}$  basis elements into the generator set.

The six bivectors  $\{e_{0i}, e_{jk}\}$  do not, of themselves, form a closed group under multiplication. This generator set introduces the scalar  $\mathbb{1}$  and the pseudo-scalar  $e_{0123}$  under multiplication. As discussed in Section 2.7, “The Duality Principle & Duality Operators”, the pseudo-scalar  $e_{0123}$  corresponds to the duality operator in four-space. The duality operator has the effect, amongst others, of transforming  $e_{0i}$  set to  $e_{ij}$  and vice versa, within a sign, (See also Section 4.5, “The Dual Field Strength Tensor”). Therefore this transformation operator can be seen as a means of switching between the rotors and the boosts. The presence of this operator



is an extended consequence of the duality principle as applied to rotations in a four dimensional spacetime. Despite being outside the classical definition of the Lorentz group, this operator must be present in the group presented here in order to satisfy the duality principle (Section 2.7, “The Duality Principle & Duality Operators”). The utility of the Lorentz group within the algebra presented here is explored in more detail in Chapter 5.

### 2.9.5 Dirac Algebras

The material contained in this section is an original re-examination of a often quoted fact[8]. An algebra for which an isomorphism can be found with the Dirac gamma matrix algebra is considered a Dirac algebra[14]. The development of the Dirac algebra commutation relations (See page 254-256 reference[14]) came about in an effort to find a first order differential form of the relativistic Schrödinger equation. The Dirac equation[33, 62, 63] :-

$$(p_0 - \alpha_i p_i - \beta) \psi = 0 \quad (96)$$

when multiplied by the conjugate operator[14]  $(p_0 + \alpha_i p_i + \beta)$ , gives: -

$$(p_0^2 - \alpha_i^2 p_i^2 + (\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + (\alpha_i \beta + \beta \alpha_i) p_i - \beta^2) \psi = 0 \quad (97)$$

which if the following relations are satisfied: -

$$\alpha_i^2 = 1 \quad , \quad \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad , \quad \beta^2 = m^2 c^2 \quad , \quad \alpha_i \beta + \beta \alpha_i = 0 \quad (98)$$

is equivalent[14] to: -

$$(p_0^2 - (m^2 c^2 + p_i^2)) \psi = 0 \quad (99)$$

If  $\beta = \alpha_m m c$  is also taken, then the relations may be summarised by the single equation:-

$$\alpha_a \alpha_b + \alpha_b \alpha_a = 2\delta_{ab} \quad (a, b = i \text{ or } m). \quad (100)$$

All four of these  $\alpha$ 's anti-commute with one another and the square of each is unity. An isomorphism may be identified between the  $\alpha_i$ 's and the spacetime

bivectors,  $e_{0i}$  as both square to unity and are orientable. Therefore, amongst other possibilities: -

$$\alpha_i = e_{0i} \tag{101}$$

The commutation relations of these elements amongst themselves are equivalent: -

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \tag{102}$$

$$e_{0i} e_{0j} + e_{0j} e_{0i} = 0 \tag{103}$$

Similarly, an isomorphism can be found for  $a_m$ . There remains (i.e excluding the  $e_{0i}$ 's) a single basis element which both squares to unity and anti-commutes with the  $e_{0i}$ 's; this element is  $e_0$ . It is clear that this mapping satisfies the relation: -

$$\alpha_i \beta + \beta \alpha_i = 0 \tag{104}$$

$$e_{0i} e_0 m c + e_0 m c e_{0i} = 0 \tag{105}$$

From these basic mappings a full isomorphism can be derived for the all 16 elements of the spacetime algebra which is closely related to the Dirac  $\gamma$ -matrix algebra[38, 16].

Spacetime	Dirac $\gamma$	Dirac $\alpha$
$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
$e_0$	$\gamma^0$	$\alpha_m$
$e_i$	$\gamma^i$	$\alpha_m \alpha_i$
$e_{0i}$	$\gamma^0 \gamma^i$	$\alpha_i$
$e_{jk}$	$\gamma^j \gamma^k$	$-\alpha_i \alpha_j$
$e_{0jk}$	$\gamma^0 \gamma^j \gamma^k$	$-\alpha_m \alpha_i \alpha_j$
$e_{123}$	$\gamma^i \gamma^j \gamma^k$	$-\alpha_m \alpha_i \alpha_j \alpha_k$
$e_{0123}$	$\gamma^0 \gamma^i \gamma^j \gamma^k$	$-\alpha_i \alpha_j \alpha_k$

**Table 2.5.** Table showing an isomorphic mapping between the spacetime algebra  $\mathcal{Cl}_{1,3}$  and the Dirac algebra (both the  $\gamma$  and  $\alpha$  representations are shown).

Therefore an isomorphism between the spacetime algebra and the Dirac algebra has been demonstrated and as a consequence the fact that the algebra presented here is a Dirac algebra has also been proven.

### 2.9.6 Discussion & Summary

The Quaternion & Dirac algebras, along with the properties of some transformation groups have been introduced in this section. Their defining characteristics have been presented and relations with the spacetime algebra have been deduced.

Isomorphic mappings that exist between sub-groups of the spacetime algebra, the Dirac and Quaternion algebras have been demonstrated. Similarly the Lorentz transformation group has been identified with the full bivector set and this has shown that there exists an extra element in the form of the pseudo-scalar  $e_{0123}$ . Arguments for the presence of this operator within the group in terms of the four-space duality principle have been presented.

Therefore the spacetime algebra, as defined in this thesis, is a real, Dirac, Minkowskian and Lorentzian algebra. The Lorentz group has also been identified. There is, however, no isomorphic mapping which yields a closed Poincaré group under multiplication in the spacetime algebra.

## 2.10 Conjugates

In this section a number of conjugates and similar operations which can be applied to multi-vector expressions in the spacetime algebra will be introduced. This section is a digest of the work of Lounesto and M.B. van der Mark[27, 38]. The discussion presented here is limited to the mechanical properties of the conjugates and how they are implemented. The discussion on the physical significance of any conjugate is deferred to the point at which it is utilised (Section 5.2, “The Lorentz Transformation”). In the spacetime algebra there are many different types of conjugation, reflection and inversion operations (a summary is available in Table 2.6), the full treatment of the which is beyond the scope of this thesis. The material provided here is for reference only[38].

Conjugate, reflection or inversion operations only reverse the signs of specific sets of basis elements; they do not change the size of any argument or the grade of

any multivector element. One such operation is the “sandwich” of a multivector by a basis element, e.g  $\Phi \rightarrow e_0\Phi e_0$ . This operation has the effect of inverting the signs of the  $e_i, e_{0i}, e_{123}$  and  $e_{0123}$  elements ( $e_0e_7e_0$  in Table 2.6).

Quaternion conjugation[38, 64] is defined as

$$q^* = (a_1 + a_2i + a_3j + a_4k)^* = a_1 - a_2i - a_3j - a_4k \quad (106)$$

This results in a sign change for each basis element represented by a 2 x 2 matrix containing i, j or k. Therefore all basis elements *except* the scalar,  $e_0, e_{123}$  and  $e_{0123}$  elements experience a change in sign ( $e_7^*$  in Table 2.6).

From the matrix representations in Section 2.6.3, “Exterior Algebra Relationships”, the transpose operation will invert the signs of the set of bivectors  $\{e_{0i}\}$  and the quadrivector  $e_{0123}$ . The  $\tilde{\Phi}$  operation is defined here as the reversal of ordering in the sub-elements within each basis element of a multivector. e.g  $e_{ij} \rightarrow e_{ji}$  etc. This operation reverses the signs of the  $e_{0i}, e_{ij}, e_{0ij}$  and  $e_{123}$  sets ( $\tilde{e}_7$  in Table 2.6).

The Hermitian conjugate is defined as the conjugate transpose[65, 66]: -

$$\Phi^\dagger = (\Phi^T)^* = e_0\tilde{\Phi}e_0 \quad (107)$$

This conjugate inverts each basis element in a multivector, with the net effect being that the signs are inverted for all basis elements which square to minus one (See Section 2.6.6, “The Quotient”). These properties of the Hermitian conjugate make it particularly useful, as it will reverse the phase development of a hypercomplex wave function (See Section 4.6, “Wave Functions”).

For an investigation of symmetry operations in the Clifford algebra see the work of Kilic *et al*[67].

## 2.11 Roots of Basis Elements

### 2.11.1 Introduction

Square root quantities of the basis elements of the Dirac  $\gamma$ -matrix algebra[35] will be examined and investigated in this section. This section is original work.

Table of signs: reflection, rotation, inversion, etc.						repr. dep.	
$\mathcal{Cl}_{1,3}$	$e_0 e_7 e_0$ $\mathcal{P}$	$e_{123} e_7 e_{kji}$ $-\mathcal{CT}$	$e_{0123} e_7 e_{kji0}$ $-\mathcal{PCT}$	$\tilde{e}_7$	$e_7^2$	$e_7^T$	$e_7^*$
$\mathbb{1}$	+	-	-	+	+	+	+
$e_0$	+	+	+	+	+	+	+
$e_i$	-	-	+	+	-	+	-
$e_{0i}$	-	+	-	-	+	-	-
$e_{ij}$	+	-	-	-	-	+	-
$e_{0ij}$	+	+	+	-	-	+	-
$e_{123}$	-	-	+	-	+	+	+
$e_{0123}$	-	+	-	+	-	-	+

**Table 2.6.** Table showing how the basis sets behave under various conjugations.

The physical motivation for finding root expressions is in their usage in relativistic quantum mechanics[14]. Roots of the nature presented here, along with their conjugates, have been employed[17, 15] in wave functions using the Clifford algebra[27, 13, 68]  $\mathcal{Cl}_{1,3}$ , which is an isomorphism of the  $\gamma$ -matrix algebra[35, 34]. No attempt will be made to interpret the physical significance of the results presented in this section. The findings are presented as a curiosity and for reference.

The aim of this section is to provide a complete reference to the  $2^{nd}$  order roots of basis elements. This begins by identifying the simplest expression, which when squared gives a particular basis element. The six elements which square to +1 are denoted generally as  $e_{\oplus}$ ,  $e_{\oplus} \in \{\mathbb{1}, e_0, e_{0i}, e_{123}\}$  and the ten elements which square to -1 as  $e_{\ominus}$ ,  $e_{\ominus} \in \{e_i, e_{jk}, e_{0jk}, e_{0123}\}$ . [35]

Euler's formula, replacing  $i$  with a member of  $e_{\ominus}$ , gives  $\exp(e_{\ominus}\phi/K) = \cos(\phi/K) + e_{\ominus} \sin(\phi/K)$  which are  $K^{th}$  roots of  $e_{\ominus}$  for  $\phi = \frac{\pi}{2}$ . When  $K = 2$ , square roots are found of  $e_{\ominus}$  with coefficients of equal magnitude, as is illustrated in Eq. (108) below.

$$\left(\frac{1}{\sqrt{2}}(1 + e_i)\right)^2 = \frac{1}{2}(1 + e_i + e_i - 1) = e_i \quad (108)$$

Eq. (108) is the most elementary expression for a square root of  $e_i$ . This can be altered, trivially, to give a root of any  $e_{\ominus}$  element by substituting  $e_i$  in Eq. (108) for the one required. There are, however, many other roots of the basis element  $e_i$ . These arise due to the way in which basis elements can combine, through

multiplication, to form other elements of different forms[35, 17]. A further root of the element  $e_i$  can be found:

$$\left(\frac{1}{\sqrt{2}}(e_{123} - e_{jk})\right)^2 = \frac{1}{2}(1 + e_i + e_i - 1) = e_i \quad (109)$$

It is therefore apparent that multiple roots to the  $\gamma$ -matrix elements exist. The following, however, is not a root of  $e_i$ .

$$\left(\frac{1}{\sqrt{2}}(e_{0i} - e_0)\right)^2 = \frac{1}{2}(1 + e_i - e_i + 1) = 1 \quad (110)$$

This is one of the roots of unity. A two term root of a  $e_\Theta$  requires one  $e_\Theta$  element and one  $e_\Phi$  element. For the case of  $e_\Theta$ , there are no two term expressions which square to a such an element alone.

### 2.11.2 Investigation

The script utilises the algebraic software which will be introduced in Chapter. 3. The algorithm used to find base roots of the  $\gamma$ -matrix elements will be presented in analytical terms here. The method requires that Eq. (111) below, is satisfied in such a way that when it can be expressed as a single term basis element

$$\Psi_{basis} = \left[ \frac{1}{\sqrt[K]{N}} (\pm\nu_1 \pm \nu_2 \dots \pm \nu_N) \right]^K \quad (111)$$

Then Eq. (112) is valid, and represents a  $K$ 'th root of that basis element.

$$\sqrt[K]{\Psi_{basis}} = \left[ \frac{1}{\sqrt[K]{N}} (\pm\nu_1 \pm \nu_2 \dots \pm \nu_N) \right] \quad (112)$$

Here the  $\nu$ 's represent all possible independent combinations of the 16 elements[35, 17, 34]. Note that the investigation is constrained, in the first instance, to where all  $\nu$ 's have coefficients of equal magnitude. Many more roots can be found if this condition is relaxed. In Eq. (112),  $N$  takes on all possible values between 1 and 16 and each sign is set independently from all others. The computer algorithm generates every possible combination of  $N$  term  $\gamma$ -matrix elements, squares them and examines the result. This test searches the entire phase space for a  $K^{\text{th}}$  root

exhaustively. The  $\frac{1}{\sqrt[N]{K}}$  factor in Eq. (111) and (112) is the standard normalisation step and is based on the number of terms in the trial solution ( $N$ ) and the power of the trial root under examination ( $K$ ).

From Eq. (111) and (112), the only element which has single term roots ( $N = 1$ ) is the scalar. This is obvious as all individual elements square to positive (six elements) or negative unity (ten elements).

### 2.11.3 Unique Roots

There are cases where a square root may be reduced to a simpler form with a lower number of terms. These special cases can be identified by introducing an extra check procedure. This check applies the  $\frac{1}{\sqrt[N]{K}}$  normalisation factor. In the case where the root is not normalised by this process it is not unique and is therefore not included in the results.

An illustrative example is where an expression contains a sub-expression which reduces to zero. In the case of the prospective root  $\Upsilon$  of  $\Psi_{basis}$  is  $e_{0ij}$

$$\Upsilon = \sqrt[2]{\Psi} = \frac{1}{\sqrt{4}}(e_0 + e_i + e_j + e_{0i}) \quad (113)$$

the applied factor does not normalise the result:

$$\Psi = \left( \frac{1}{\sqrt{4}}(e_0 + e_i + e_j + e_{0i}) \right)^2 = \frac{1}{2}e_{0ij} = \frac{1}{2}\Psi_{basis} \quad (114)$$

because the root expression can be reduced to a simpler form. The  $e_0 + e_i$  part of the expression vanishes when squared, disguising the fact that the expression is essentially a two term root. This vanishing part is a so-called “null-vector”, a vector of squared length zero, which may be written more generally as

$$\Upsilon_{null}^2 = (e_{\oplus} + e_{\ominus})^2 = 0 \quad \text{where } \{e_{\oplus}, e_{\ominus}\} = 0 \quad (115)$$

$\Upsilon_{null}$  should be compared to a zero-interval four-vector or an electromagnetic null-vector as discussed by Kramers[69], though here both the non-commuting nature of the algebra and the metric signature play a role. Discrepancies in the normalisation stage allow roots such as Eq. (113) to be identified and filtered. The results presented in this section contain only roots which normalise using a

$\Psi_{basis}$	$\Upsilon$	$\Psi_{basis}$	$\Upsilon$
$e_i$	$\pm \frac{1}{\sqrt{2}}(1 + e_i)$	$-e_i$	$\pm \frac{1}{\sqrt{2}}(1 - e_i)$
$e_i$	$\pm \frac{1}{\sqrt{2}}(e_{0j} - e_{0ij})$	$-e_i$	$\pm \frac{1}{\sqrt{2}}(e_{0j} + e_{0ij})$
$e_i$	$\pm \frac{1}{\sqrt{2}}(e_{0k} + e_{0ki})$	$-e_i$	$\pm \frac{1}{\sqrt{2}}(e_{0k} - e_{0ki})$
$e_i$	$\pm \frac{1}{\sqrt{2}}(e_{jk} - e_{123})$	$-e_i$	$\pm \frac{1}{\sqrt{2}}(e_{jk} + e_{123})$
$e_{jk}$	$\pm \frac{1}{\sqrt{2}}(1 + e_{jk})$	$-e_{jk}$	$\pm \frac{1}{\sqrt{2}}(1 - e_{jk})$
$e_{jk}$	$\pm \frac{1}{\sqrt{2}}(e_0 + e_{0jk})$	$-e_{jk}$	$\pm \frac{1}{\sqrt{2}}(e_0 - e_{0jk})$
$e_{jk}$	$\pm \frac{1}{\sqrt{2}}(e_i - e_{123})$	$-e_{jk}$	$\pm \frac{1}{\sqrt{2}}(e_i + e_{123})$
$e_{jk}$	$\pm \frac{1}{\sqrt{2}}(e_{0i} + e_{0123})$	$-e_{jk}$	$\pm \frac{1}{\sqrt{2}}(e_{0i} - e_{0123})$
$e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(1 + e_{0jk})$	$-e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(1 - e_{0jk})$
$e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(e_k + e_{0j})$	$-e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(e_k - e_{0j})$
$e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(e_j - e_{0k})$	$-e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(e_j + e_{0k})$
$e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(e_0 + e_{jk})$	$-e_{0jk}$	$\pm \frac{1}{\sqrt{2}}(e_0 - e_{jk})$
$e_{0123}$	$\pm \frac{1}{\sqrt{2}}(1 + e_{0123})$	$-e_{0123}$	$\pm \frac{1}{\sqrt{2}}(1 - e_{0123})$
$e_{0123}$	$\pm \frac{1}{\sqrt{2}}(e_{0i} + e_{jk})$	$-e_{0123}$	$\pm \frac{1}{\sqrt{2}}(e_{0i} - e_{jk})$

**Table 2.7.** Table listing positive and negative two term roots of basis elements. Cyclic permutations apply to give all possible roots.

factor of  $\frac{1}{\sqrt{N}}$ . In the context of this section such roots are referred to as unique roots.

### 2.11.4 Square Roots

The roots of the scalar, both positive and negative, are formed from sums of anti-commuting elements. These sums must consist of only  $e_{\oplus}$  or  $e_{\ominus}$  elements to produce unique roots of the positive and negative scalar respectively:

$$\left( \frac{1}{\sqrt{N}} \sum_{n=1}^N e_{\oplus} \right)^2 = 1, \quad \left( \frac{1}{\sqrt{N}} \sum_{n=1}^N e_{\ominus} \right)^2 = -1 \quad (116)$$

Here each element of the sums in Eq. (116) anti-commutes with *all* others. Mixing anti-commuting sums of  $e_{\oplus}$  and  $e_{\ominus}$  yields a scalar; however the root is not unique in the sense that the factor  $\frac{1}{\sqrt{N}}$  does not normalise the result. The contributions of more  $e_{\oplus}$  than  $e_{\ominus}$  in the sum gives rise to a positive scalar and vice versa. Due to their abundance the roots of the scalar are not presented in tabulated form.

All, non-scalar, two term roots are presented in Table. 2.7. These roots can



also be represented by:

$$\left( \frac{1}{\sqrt{2}} e_P (1 \pm e_P^2 e_\Theta) \right)^2 = \pm e_\Theta \quad \text{where } [e_P, e_\Theta] = 0 \quad (117)$$

The choice of  $\pm$  on the left hand side yields that sign on the right. Eq. (117) applies to a specific  $e_\Theta$  and any  $e_P$  which commutes with it, and  $e_P$  can be any element (including the identity  $\mathbb{1}$ ) of the  $\gamma$ -algebra as long as the commutation relation is fulfilled. The index  $P$  is some ‘‘hyper index’’ which can, for example, take on the values  $\mu$ , but also  $ij$ , or  $0123$ .

There are no two term roots of the  $e_\Theta$  elements  $\{e_0, e_{0i}, e_{123}\}$ . In the two term case, each  $e_\Theta$  element has eight positive roots and eight negative roots. There are no unique roots for three terms cases ( $N = 3$ ) with coefficients of equal magnitude. Such odd number terms, when squared, will give three scalar components of unit magnitude. A scalar component will, therefore, always be present. This applies to all such expressions with an odd number of linearly independent terms.

The four term roots can be represented by the following equations:

$$\left( \frac{1}{2} (1 \pm e_\oplus + e_\ominus \mp e_\ominus e_\oplus) \right)^2 = \pm e_\oplus \quad (118)$$

where  $[e_\ominus, e_\oplus] = 0$

and

$$\left( \frac{1}{2} (e_P + e_Q \pm (e_P^3 + e_Q^3) e_\Theta) \right)^2 = \pm e_\Theta \quad (119)$$

where  $[e_P, e_\Theta] = [e_Q, e_\Theta] = 0, \{e_P, e_Q\} = 0$

The square roots which comprise of four terms and six terms ( $N = 4$  and  $N = 6$ ) are presented in Tables 2.8 and 2.9 respectively. As for the case of two term roots, there are the same number of unique positive and negative roots for each of the  $\gamma$ -matrix elements present. In the case of four term roots there are twelve positive and twelve negative square roots. Note that unlike the two and six term case, in the four term case ( $N = 4$ ) there are the same number of roots present for each of the  $e_\Theta$  elements as there are for the  $e_\oplus$  elements.

$\Psi_{basis}$	$\Upsilon$	$\Psi_{basis}$	$\Upsilon$
$e_0$	$\frac{1}{2}(1 + e_0 \pm e_{ij} \mp e_{0ij})$	$-e_0$	$\frac{1}{2}(1 - e_0 \pm e_{ij} \pm e_{0ij})$
$e_i$	$\frac{1}{2}(e_{0j} \pm e_{0k} - e_{0ij} \pm e_{0ki})$	$-e_i$	$\frac{1}{2}(e_{0j} \pm e_{0k} + e_{0ij} \mp e_{0ki})$
$e_i$	$\frac{1}{2}(e_{0j} \pm e_{jk} - e_{0ij} \mp e_{123})$	$-e_i$	$\frac{1}{2}(e_{0j} \pm e_{jk} + e_{0ij} \mp e_{123})$
$e_i$	$\frac{1}{2}(e_{0k} \pm e_{jk} + e_{0ki} \mp e_{123})$	$-e_i$	$\frac{1}{2}(e_{0k} \pm e_{jk} - e_{0ki} \pm e_{123})$
$e_{0i}$	$\frac{1}{2}(1 \pm e_j + e_{0i} \mp e_{0ij})$	$-e_{0i}$	$\frac{1}{2}(1 \pm e_j - e_{0i} \pm e_{0ij})$
$e_{0i}$	$\frac{1}{2}(1 \pm e_k + e_{0i} \pm e_{0ki})$	$-e_{0i}$	$\frac{1}{2}(1 \pm e_k - e_{0i} \mp e_{0ki})$
$e_{0i}$	$\frac{1}{2}(1 + e_{0i} \pm e_{jk} \mp e_{0123})$	$-e_{0i}$	$\frac{1}{2}(1 - e_{0i} \pm e_{jk} \pm e_{0123})$
$e_{jk}$	$\frac{1}{2}(e_0 \pm e_i + e_{0jk} \mp e_{123})$	$-e_{jk}$	$\frac{1}{2}(e_0 \pm e_i - e_{0jk} \pm e_{123})$
$e_{jk}$	$\frac{1}{2}(e_0 \pm e_{0i} + e_{0jk} \pm e_{0123})$	$-e_{jk}$	$\frac{1}{2}(e_0 \pm e_{0i} - e_{0jk} \mp e_{0123})$
$e_{jk}$	$\frac{1}{2}(e_i \pm e_{0i} - e_{123} \pm e_{0123})$	$-e_{jk}$	$\frac{1}{2}(e_i \pm e_{0i} + e_{123} \mp e_{0123})$
$e_{0jk}$	$\frac{1}{2}(e_j \pm e_k \pm e_{0j} - e_{0k})$	$-e_{0jk}$	$\frac{1}{2}(e_j \pm e_k \mp e_{0j} + e_{0k})$
$e_{0jk}$	$\frac{1}{2}(e_0 \pm e_k \pm e_{0j} + e_{jk})$	$-e_{0jk}$	$\frac{1}{2}(e_0 \pm e_k \mp e_{0j} - e_{jk})$
$e_{0jk}$	$\frac{1}{2}(e_0 \pm e_j \mp e_{0k} + e_{jk})$	$-e_{0jk}$	$\frac{1}{2}(e_0 \pm e_j \pm e_{0k} - e_{jk})$
$e_{123}$	$\frac{1}{2}(1 \pm e_i \pm e_{jk} + e_{123})$	$-e_{123}$	$\frac{1}{2}(1 \pm e_i \mp e_{jk} - e_{123})$
$e_{0123}$	$\frac{1}{2}(e_{0i} \pm e_{0j} + e_{jk} \pm e_{ki})$	$-e_{0123}$	$\frac{1}{2}(e_{0i} \pm e_{0j} - e_{jk} \mp e_{ki})$

**Table 2.8.** Table listing positive and negative four term roots of basis elements. Cyclic permutations apply to give all possible roots. Each  $\pm$  within a term is not independent. For any term where  $\pm$  is set, it must be similarly set across the term.

All the four and six term roots of  $e_\Theta$  elements present in Tables 2.8 and 2.9 are comprised of sums of linearly independent two term roots of the same element. None of these roots contain the scalar or  $e_\Theta$  element itself. This will be examined later.

The six term table (Table 2.9) has entries for all members of  $e_\Theta$ . Note that there are fewer six term unique roots for each element, when compared to the case of two and four terms. An analytical form of Table 2.9 is found by extending the form of Eq. (119) to six terms as seen in Eq. (120). The algorithm returned no unique square root expressions comprising of more than six terms ( $N > 6$ ).

$$\left( \frac{1}{\sqrt{6}} (e_P + e_Q + e_R \pm (e_P^3 + e_Q^3 + e_R^3)e_\Theta) \right)^2 = \pm e_\Theta \quad (120)$$

where  $[e_P, e_\Theta] = [e_Q, e_\Theta] = [e_R, e_\Theta] = 0$   
and  $\{e_P, e_Q\} = \{e_Q, e_R\} = \{e_R, e_P\} = 0$

$\Psi$ basis	$\Upsilon$
$e_i$	$\frac{1}{\sqrt{6}}(e_{0j} - e_{0k} \pm e_{jk} - e_{0ij} - e_{0ki} \mp e_{123})$
$e_i$	$\frac{1}{\sqrt{6}}(e_{0j} + e_{0k} \pm e_{jk} - e_{0ij} + e_{0ki} \mp e_{123})$
$-e_i$	$\frac{1}{\sqrt{6}}(e_{0j} + e_{0k} \pm e_{jk} + e_{0ij} - e_{0ki} \pm e_{123})$
$-e_i$	$\frac{1}{\sqrt{6}}(e_{0j} - e_{0k} \pm e_{jk} + e_{0ij} + e_{0ki} \pm e_{123})$
$e_{jk}$	$\frac{1}{\sqrt{6}}(e_0 + e_i \pm e_{0i} + e_{0jk} - e_{123} \pm e_{0123})$
$e_{jk}$	$\frac{1}{\sqrt{6}}(e_0 - e_i \pm e_{0i} + e_{0jk} + e_{123} \pm e_{0123})$
$-e_{jk}$	$\frac{1}{\sqrt{6}}(e_0 - e_i \pm e_{0i} - e_{0jk} - e_{123} \mp e_{0123})$
$-e_{jk}$	$\frac{1}{\sqrt{6}}(e_0 + e_i \pm e_{0i} - e_{0jk} + e_{123} \mp e_{0123})$
$e_{0jk}$	$\frac{1}{\sqrt{6}}(e_0 \pm e_j - e_k - e_{0j} \mp e_{0k} + e_{jk})$
$e_{0jk}$	$\frac{1}{\sqrt{6}}(e_0 \pm e_j + e_k + e_{0j} \mp e_{0k} + e_{jk})$
$-e_{0jk}$	$\frac{1}{\sqrt{6}}(e_0 \pm e_j + e_k - e_{0j} \pm e_{0k} - e_{jk})$
$-e_{0jk}$	$\frac{1}{\sqrt{6}}(e_0 \pm e_j - e_k + e_{0j} \pm e_{0k} - e_{jk})$
$e_{0123}$	$\frac{1}{\sqrt{6}}(\pm e_{0i} + e_{0j} + e_{0k} \pm e_{jk} + e_{ki} + e_{ij})$
$e_{0123}$	$\frac{1}{\sqrt{6}}(\pm e_{0i} - e_{0j} + e_{0k} \pm e_{jk} - e_{ki} + e_{ij})$
$-e_{0123}$	$\frac{1}{\sqrt{6}}(\pm e_{0i} + e_{0j} + e_{0k} \mp e_{jk} - e_{ki} - e_{ij})$
$-e_{0123}$	$\frac{1}{\sqrt{6}}(\pm e_{0i} - e_{0j} + e_{0k} \mp e_{jk} + e_{ki} - e_{ij})$

**Table 2.9.** Table listing positive and negative six term roots of basis elements. Cyclic permutations apply to give all possible roots. Each  $\pm$  within a term is not independent. For any term where  $\pm$  is set, it must be similarly set across the term.

### 2.11.5 Analysis

Let  $e_S$  be the basis element for which a root is to be found. The sets of  $e_\oplus$  and  $e_\ominus$  which commute with  $e_S$  are defined as follows:

$$e_{\bar{\oplus}S} = \{e_\oplus : [e_S, e_\oplus] = 0\} \quad (121)$$

$$e_{\bar{\ominus}S} = \{e_\ominus : [e_S, e_\ominus] = 0\} \quad (122)$$

From the results in Tables 2.7, 2.8 and 2.9, a fully expanded unique root will contain only  $e_{\bar{\oplus}S}$  and  $e_{\bar{\ominus}S}$  elements for a given  $e_S$ . Within this expanded sum there will be a  $e_{\bar{\ominus}S}$  element for each  $e_{\bar{\oplus}S}$  member present, where  $e_S$  is a member of  $e_\oplus$ ,  $e_{\bar{\oplus}S} = \{1, e_S\}$ . Thus members of  $e_\oplus$  are present in their own roots. Roots of such elements have exactly four terms as the size of the  $e_{\bar{\oplus}S}$  set is maximally two. In other words the requirement for roots of members of  $e_\oplus$  to have exactly four terms comes from the fact that these elements anti-commute with all other  $e_\oplus$  elements with the exception of the scalar.

The four and six term  $e_\ominus$  roots comprise of sums of two term roots. Let the elementary two-term root which squares to  $\pm e_\ominus$  be  $\Upsilon_{\pm e_\ominus}^{\mathbb{1}} = (1 \pm e_\ominus)/\sqrt{2}$  and let

the three remaining positive and negative two-term cases be denoted  $\Upsilon_{\pm e_{\Theta}}^{e_P}$ :

$$\Upsilon_{\pm e_{\Theta}}^{e_P} = \frac{1}{\sqrt{2}} e_P (1 \pm e_P^2 e_{\Theta}) = \sqrt{\pm e_{\Theta}} \quad (123)$$

where  $[e_P, e_{\Theta}] = 0$

Note that these three linearly independent roots are qualitatively different to the elementary root since they do not have an analogous form to the Euler formula, and hence do not readily generalise to  $K^{\text{th}}$ -order roots. They do, however, have the property that any normalised linear combination of them is also a root of the element under consideration; that is the four and six term roots of the  $e_{\Theta}$ -set are all linear combinations of the corresponding two-term roots. A generalised four or six term root is : -

$$\left( \frac{a\Upsilon_{\pm e_{\Theta}}^{e_P} + b\Upsilon_{\pm e_{\Theta}}^{e_Q} + c\Upsilon_{\pm e_{\Theta}}^{e_R}}{\sqrt{a^2 + b^2 + c^2}} \right)^2 = \pm e_{\Theta} \quad (124)$$

In Eq. (124) the square of the sums gives the sum of the squares. Note that the (anti-)commutation rules of Eq. (120) apply and that  $a$ ,  $b$  and  $c$  are strictly scalars and the summation terms must all be  $\Upsilon_{+e_{\Theta}}$  or all  $\Upsilon_{-e_{\Theta}}$ . The sum of one  $\Upsilon_{+e_{\Theta}}$  and one  $\Upsilon_{-e_{\Theta}}$  will give zero when squared. They must be linearly independent of each other for the cross terms to vanish.

### 2.11.6 Discussion

The methods presented in this section have been used to investigate the conditions under which a root can be found. In the case of two term expressions ( $N = 2$ ), there are eight positive and eight negative roots for each element which square to minus one. There are, however, no two term square roots for a  $e_{\oplus}$  element. This is also the case for six term expressions. The results show that for four terms, there are multiple unique (positive and negative) roots for all basis elements. This is the only number of terms for which this is true.

The fact that the four and six term roots of  $e_{\Theta}$  are comprised of sums of the non-elementary two term roots means these elements must be linearly independent of the  $e_{\Theta}$  element and the scalar in order for the sum to be a root. As there

are only three such two term roots for each  $e_{\ominus}$  element the maximum number of terms possible for a root is six. This is also confirmed by the brute force search.

Each element of the algebra, aside from the scalar, commutes with eight other elements. In the case of the  $e_{\ominus}$  elements, each commutes with four members of  $e_{\oplus}$  and four members of  $e_{\ominus}$ . As all unique roots of basis elements are comprised of equal numbers of  $e_{\oplus}$  and  $e_{\ominus}$  elements, unique roots of this form can only exist for up to eight terms.

## 2.12 Chapter Summary

In this chapter Clifford algebras and the Spacetime algebra  $\mathcal{Cl}_{1,3}$  have been presented and examined. The differences between the Clifford product and the standard cross product has been shown. The basis elements of Clifford algebras have been introduced and represented geometrically. Bivectors, trivectors and quadrivectors have also been introduced. The differential operator  $d$  has been derived from the quotient relations of vectors. The vector nature of this operator has been shown to give rise to a change of grade when applied.

A number of closed groups under multiplication have been identified. Isomorphisms between these groups and complex numbers and the even-subalgebra used by Gull *et al*[17] have been presented. By investigation of it's properties, the spacetime algebra  $\mathcal{Cl}_{1,3}$  has been shown to be a real, Dirac, Minkowskian and Lorentzian algebra. A sub-group of the algebra has been identified as isomorphic with the Quaternion group. The Lorentz group has also been identified. There is, however, no isomorphic mapping which yields a closed Poincaré group under multiplication in the spacetime algebra.

The concept of the duality operator has also been introduced and explained. The Hodge dual[27] was introduced, firstly in the three dimensional Euclidean metric, in terms of a *duality principle*[50]. A description of the duality principle in the four dimensional Lorentz metric was then presented. This operation, along with the space-time split[34], have been utilised as a means of assigning “parallelness” and perpendicularity between various basis elements. Eq. (82), (83) & (84) can be used as reference for such attributes. Applications of this table can be found in Chapter 5.

In this chapter some simple as well as some brute force methods for determining roots of  $\gamma$ -matrix elements have been presented. The investigation was constrained to the study of square roots with coefficients of equal magnitude; however these techniques and methods may be applied to higher order roots.

A mechanism for identifying repeating and non-unique roots has been developed which examines the normalisation of the root when squared. It has been shown that unique roots of non-scalar basis elements must contain an even number of linearly independent terms. The simplest roots of the  $e_{\oplus}$  elements have two terms, where the simplest roots of the  $e_{\ominus}$  elements have four terms. The square of the sums of linearly independent roots of the  $e_{\oplus}$  elements have been shown to be equal to the sum of their squares.

Each  $e_{\oplus}$  element commutes with four  $e_{\oplus}$  and four  $e_{\ominus}$  elements, limiting roots of this form to an absolute maximum of eight terms. This is confirmed by the root search algorithm. This check has been conducted for all cases up to that of twelve terms.

The requirement for roots of members of  $e_{\oplus}$  to have exactly four terms has been demonstrated. No unique roots are found where the number of terms exceeds six; leading to the conclusion that all unique roots have been found by this point.

The motivation for investigating electromagnetism in the algebra presented in this chapter is that it is a four dimensional Dirac-Minkowski algebra for which an isomorphism with the Quaternion algebra can be found. Further motivation is that the algebra describes rotations correctly, does not require the introduction of the unit imaginary  $i$  and has a real invariant interval.

# Chapter 3

## Algebraic Software

### 3.1 Introduction

A Computer Algebra System is a program that can be used to do symbolic manipulation of mathematical expressions. It is built on top of a programming language designed for this purpose, in which new algorithms can easily be implemented. Such computer based programs are capable of representing, performing calculations and manipulating algebraic expressions in various algebras. This chapter is concerned with the development of software capable of such operations in the Clifford algebra  $Cl_{1,3}$ [13, 17, 26] (See Section 2.6, “The Spacetime Algebra”). The motivation for developing a software based system is due to the tediousness involved in calculating the large number of terms which appear for products in non-Abelian four dimensional algebras. Mistakes are easily made by humans in these situations, and even one error in the signs can lead to an incorrect and unusable result; as many terms which should cancel in the higher orders will not do so (See Section 2.6.4, “Higher Grade Objects”). Enabling the possibility of scripting of large phase space searches for patterns and symmetries in the algebra is a further motivation.

The aim of this Chapter is to present the development, usage and applications of the algebraic software. This chapter contains completely original work. The requirements will be introduced and discussed and a review of some pre-existing (third party) packages capable of manipulating geometric algebras will be conducted. Although there is no desire or intention to “re-invent the wheel”

in this chapter it is paramount that any software developed exhibits the feature set required for the research presented in this thesis.

The specification for the software development will be presented in this chapter. The choice of programming language, along with the overall code philosophy will be put forth and discussed. This will be followed by details of all the algorithms developed. Possible alternative approaches will be discussed, accompanied with arguments for not pursuing them. Efforts and development strategies which failed to produce any results will be discussed, as well as those which were abandoned because of over complication.

An introduction of scripting, further optimisation stages and the utilisation of the final application will also be given. In addition, examples of how many common operations, such as wave functions, “proper” differential operators and instances of basis elements are created and used will be shown. Similarly, the grouping of many algebraic operations in a sequence known as a script which can be executed in the same way as a computer program will be presented. The utility of these facilities will be discussed. Finally, Numerical evaluation and dimensional separation of algebraic expressions will also be presented.

## 3.2 Algebraic Software

### 3.2.1 Introduction

In this section several projects which facilitate symbolic manipulation of Geometric/Clifford algebras will be assessed[13, 68]. These projects have been in on going development during the course of this research project. Many of the required features, which were absent from these projects when development commenced, have been incorporated over time. Such is the nature of software development. In this section parallel feature development, along with the benefits and drawbacks associated with each project, will be discussed.

The availability of the source-code and the degree of support provided with each project is taken into consideration. Source-code availability has consequences with regard to further modifications and bug-fixes should the original maintainers cease development.



### 3.2.2 CLICAL

CLICAL is a Complex Number, Vector Space and Clifford Algebra Calculator for MS-DOS Personal Computers and was developed by P. Lounesto, R. Mikkola, V. Vierros at The Institute of Mathematics, Helsinki University of Technology. The program is stand-alone and does not require any other software in order to function. The program was written in Microsoft Quick Pascal, although the source-code is not made available\*. The program can only be run in MS-DOS, and cannot therefore be run on larger mainframe computers. The lack of an open-source license also prevents the modification and maintenance of the software.

The program does however permit the saving and loading of sessions and supports scripting; which is vital to running automated simulations and equation solvers. CLICAL is a powerful tool for evaluating geometric problems. In particular, the ability to specify the number of dimensions (n dimensions are supported) and the preservation of the product of two basis vectors as a bivector are excellent features. The software handles the commutation relations of the basis elements without flaw, and supports multiple power series functions. It does not however, support the introduction of arbitrary variables in equations, or implement a differential operator; but does, however, provide for lambda functions[70]. The program functions more as a calculator, rather than an algebraic manipulation tool.

A major downside with CLICAL is that there is no provision made for an arbitrary choice of metric. The program assumes that the square of any basis element is +1 (i.e. It uses the (+ + + +) metric in 4 space), and that the square root of -1 is taken to be  $i$ .

### 3.2.3 Geometric Calculator

The Geometric Calculator was developed by Roger. E. Critchlow Jr, and is available from <http://www.elf.org/calculator>. The application is *an ordinary desk calculator that uses the Clifford Numbers over a three dimensional Euclidean Space*. The program is a Java based, and is therefore completely hardware/os independent. The human interface is exceedingly well designed and implemented

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\*<http://users.tkk.fi/u/ppuska/mirror/Lounesto/CLICAL.htm>

in this case. The documentation is also available in an online form, along with tutorials and examples, thus making this application an excellent choice for teaching purposes. This program does a rather good job for its intended purpose. The application is stable and does exactly what it claims to be able to do. It correctly preserves the multiplication of two unit vectors as a compound object and contains the correct commutation relations.

Unfortunately this application does not have many of the required features. The software lacks facilities for algebraic manipulation, differentiation, wave functions and there is no choice of metric tensor. The metric tensor is limited, to the (+ + +) metric. Despite the lack of required functionality for the applications presented here, The Geometric Calculator is a still highly recommended piece of software.

### 3.2.4 Discussion & Summary

In this section some packages for manipulating Clifford algebras have been examined and assessed. Unfortunately none of the available packages meet the basic requirements here. In most cases the lack of support for different metric tensors is the most crucial factor. The lack, in most cases, of support for the multivector differential operator, is the crucial factor. Add-on packages for Mathematica, Maple and MathCAD had the same failings. This facility is, as discussed, required for the verification of the solutions to differential equations.

## 3.3 Software Development

### 3.3.1 Introduction

In this section the specification for the algebraic software will be introduced. Details of the capabilities of each aspect of the feature set will be detailed and where appropriate discussions will be presented. Similarly, the object oriented approach to the programming structure will be introduced in this section. The top down hierarchical structure will be presented and discussed.

The spacetime reduction algorithm will also be presented in this section. This

is a method for taking two spacetime basis elements (See Section 2.6, “The Spacetime Algebra”) and combining them as one, using the commutation and metric properties of the Clifford algebra  $\mathcal{C}\ell_{1,3}$ . These rules flow precisely from the way manual reduction such products are performed. This algorithm will be shown to be a reliable mechanical means for automating this task. This takes care of both the metric tensor and the commutative rules of the algebra in one procedure.

### 3.3.2 Specification

The algebraic software must be able to manipulate, expand and differentiate equations in the Clifford algebra  $\mathcal{C}\ell_{1,3}$ . (See Section 2.6, “The Spacetime Algebra”). The software must be capable of performing the algebraic reduction of basis elements (See Section 2.4, “Basis Elements”), must support the Lorentz metric (+ - - -), the anti-Lorentz metric (- + + +) and the Euclidean metric (+ + +). There must make a clear distinction between these elements through the use different notations. Steps must be taken to ensure that the order of multiplication is always rigorously preserved. Furthermore, any reduction algorithms developed or used to reduce expressions must be able to do so whilst preserving the commutation relations of the algebra.

Scripting of algebraic operations is also required. This must allow a number of operations to be performed as one, and then be re-run at a later date if required. This is to provide repeatability in results. The scripting mechanism must be sufficiently intuitive that a person with little programming experience can use and understand it relatively easily.

The software must be able to perform differentiation (integration is not required) and those operations should be customisable to allow for transformations, manipulations in different coordinate systems and “grades” of the differential operator. The differential operator implementation is required to allow one to ascertain whether a function is a solution to a homogeneous wave equation. Thus permitting the examination of a great many potential solutions and enabling one to rapidly eliminate those which are non-physical. Thus, a larger phase space of potential solutions can be examined. As a consequence, wave equations must be supported within the algebraic framework and appropriate routines must be available for visualisation of those wave equations.

Speed of execution, whilst a consideration, is not of paramount concern here. Evidence to demonstrate the any steps taken to optimise process execution must be provided to ensure that accuracy of results are never compromised over speed of execution.

### 3.3.3 Programming Language Choice

The programming languages under consideration for the implementation of the algebraic software are C, C++, FORTRAN, Lisp and Python. The next section details the relative advantages and disadvantages of each.

C++ and C are the first prospective language choice which will be examined. This is primarily because the language is the lowest level of programming language available which still permits ease of portability across the various different computer platforms available. As a result code where speed is paramount is usually developed in these languages. Programming software which manipulates strings in C and C++ can, however, be both tedious and difficult to debug. The development time would be effectively doubled in choosing this programming language over other higher level languages available. C/C++ is also far harder to learn and document, and therefore future developers would find it much more difficult to determine how a particular piece of code functions compared to higher level programming languages. It may however, be possible to utilise C and/or C++ for implementing libraries which other languages could utilise to speed up execution of critical portions of the code.

The next language considered here is FORTRAN\*. FORTRAN was developed by a team of programmers at IBM, and was first published in 1957. The name FORTRAN is an acronym for FORMula TRANslation, because it is designed to allow easy translation of math formulas into code. FORTRAN is also the first ever high-level language, using the first compiler ever developed. The objective during it's design was to create a programming language that would be: simple to learn, suitable for a wide variety of applications, machine independent, and would allow complex mathematical expressions to be stated similarly to regular algebraic notation, while still being almost as efficient in execution as assembly

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\*Parts of this discussion are taken from <http://www.engin.umd.umich.edu/CIS/course.des/cis400/fortran/fortran.html>\hash{}history

language. Since FORTRAN is so much easier to code, programmers were able to write programs 500% faster than before, while execution efficiency is only reduced by 20%, this allowed them to focus more on the problem solving aspects of a problem, and less on coding. FORTRAN is an easy language to learn when compared to assembly language and is marginally easier to learn than C. Like C and all the languages presented here, the FORTRAN language is cross platform and machine-independent. However, FORTRAN as a language is geared more for numerically oriented problems rather than algebraic manipulation techniques. Strings in FORTRAN have similar issues as those in C/C++. FORTRAN is fast and efficient at processing mathematical equations, but not manipulating and implementing hypercomplex algebra reduction techniques.

A further possible choice of programming language is Lisp\*. In many ways Lisp is an ideal language choice for this particular implementation. Lisp is based on the Lambda Calculus by A. Church[70] and is designed primarily for symbolic computation. The language supports garbage collection[71], is modular, has object orientated flavours[72]. Unlike all the other programming languages discussed in this section, Lisp is a functional programming language rather than a procedural programming language. In this programming construct, symbols (atoms) are the basic data type; these symbols are put together to create structures, usually in the form of a symbolic expression, or s-expression, making this language ideal for algebraic implementations. However, functional programming languages differ greatly from the conventional procedural languages. As a result, functional programming languages such as Lisp are far more difficult to master, debug and to document sufficiently for future developers. Therefore, in an effort to reduce the complexity of an already complex project and to provide an easier learning curve for users, Lisp was not chosen as the implementation language.

A further possibility as a choice for programming language is Python[73]. Python is a clear and powerful object-oriented programming language which uses an elegant syntax, providing comparatively easily readable and understandable programs when compared to other procedural programming languages. Python is an interpreted language, supporting modules and therefore making a scripting

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\*Some of the material mentioned here is taken or adapted from <http://www.cs.wisc.edu/~cs540-1/notes/lisp.html>

implementation simple. Python does, however, support being compiled into a byte-code format which can then be interpreted much faster than the raw script, similar in some ways to the Java VM[74]. Python compiles this byte code during execution. It does not compile code in a true “Just In Time” manner, but the code is compiled at run-time and therefore a procedure or sub-routine is only interpreted once, thereafter it is executed from the byte-code version (unless it is altered in the raw script). This process can be kept completely transparent from the user and as a result, Python has all the advantages of an interpreted language, but by utilising this compilation method the execute speed problems normally associated with interpreted languages are somewhat assuaged.

A further advantage of the Python interpreter is that it is easily extendable, by adding new modules implemented in a compiled language such as C or C++. Thus allowing the optimisation of processor intensive portions of the code by utilising a lower level language. The language can support fall back, in that where an optimised or compiled module exists it can use it, otherwise it will utilise its own interpreted (slower) version. Thus allowing greater portability. In addition to this, the Python documentation and API reference manuals are superb and freely available[73].

Python has an interactive console mode, which makes testing short sections of code, importing modules and performing calculations very easy. This interactive mode behaves in many ways like Maple or MatLab. This particular feature of Python is the final justification for using this language. By utilising this console mode as the user interface for the algebraic software, there is no need to write and debug a complicated and time consuming command line (console) parser; as these are already embedded in Python and any bugs found in these parsers will be addressed by the Python developers rather than any future development team. In doing this, the overall code becomes much simpler, faster and more reliable in one step, as Python objects can be utilised as algebraic symbols or constructs. Python also supports the definition of Lambda functions, which allow the introduction of functional programming constructs if required[73].

Data types in Python are strongly but dynamically typed. Mixing incompatible types (e.g. attempting to add a string and a number) causes an exception to be raised; an exception handler can be used to force a valid result where (e.g.

transform number into a string, concatenate and return a string instead of a number.). Due to the ease of code readability, object-oriented system, availability of C/C++ library interfaces and the command-line interpreter Python is the primary programming language choice here. This choice makes future development of the software simpler and easier to debug.

### 3.3.4 Object Oriented Approach

A hierarchical object orientated approach is used to represent the basis elements, functions, numerical values and variables as objects. Each of these objects are of themselves capable of performing the required reduction, expansion and differentiation operations on themselves. This approach has an elegant simplicity to it, in that enables modular expansion of the core system as required. This approach is extremely simple to debug as faulty modules can be readily identified and repaired.

Each object in this system is an instance of a class. A class can implement certain operations that are invoked with special syntax (such as arithmetic operations or subscripting and slicing[73]) by defining methods with special names\*. This is Python's approach to operator overloading[73], allowing classes to define their own behaviour with respect to language operators. As discussed in Section 3.3.3, Python can be utilised in a command line mode, and the interpreter can be utilised to parse user entered data and dissect this information into various objects. Each object can then be called appropriately with a second object as a parameter. For example, in the case where "A \* B" is the user data, object A's (definable) multiplication handler is called with object B as a parameter, the object returned from this call is a new object which represents "A \* B", which is completely definable. Not only does this allow objects to determine how they interact with one another separately, it also rigidly preserves the order of multiplication. This is because the first object is always called with the second as a parameter.

The structured approach does require at least two container objects or classes, a term container and an expression container. Both of these object are capable

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\*These names, and some of the content here is taken from the Python documentation, found at <http://docs.python.org/ref/specialnames.html>

of handling any possible (algebraic or numeric) object. In the context of this program structure, terms consist of ordered sets of algebraic and numeric elements held in a list. There is an implied multiplication operation between elements held in the list and the order of the list is the order of multiplication. Division is achieved by setting the power attribute of an object to -1, and is handled automatically by the interface Python provides. Similarly the expression class may hold algebraic elements, numeric elements and term objects. The expression class has an implied addition operation between objects and also holds objects in a list. Like the case for division, subtraction is achieved by the negation of an object before placing it in the list. The techniques used for the reduction and expansion of terms and expressions will be discussed later.

The object hierarchy is such that the primary object is an expression. The expression class has no information about classes below it in the hierarchy and cannot generate objects of any type other than expression objects and scalar numeric values. The next object in this hierarchy is the term class. Similarly, the term class has no knowledge of any objects below it in the hierarchy but *does* have assumed knowledge of expression classes. A term class can generate expressions if required, thus allowing the term  $A(B + C)$  to be expanded automatically as  $AB + AC$  as required for our algebra.

Any other desired objects are implemented below the term and expression classes in the hierarchy. It is worth stressing at this point, that objects which are high up in the class hierarchy have no ability to create instances of objects lower than them. Objects lower down in the hierarchy will always have the ability to create instances of classes higher up in the hierarchy, and may in special cases, have the ability to create ones at the same level.

### 3.3.5 Basis Element Reduction

The basis element reduction algorithm functions by taking two elements, each represented by a string and reducing them to a single condensed string. The crucial information about a basis element are the subscript indices (e.g. '12' for a bivector  $e_{12}$ ) and the overall sign of the object. For example, the string representation of  $e_{12}$  would be "+12".

Parsers are designed such that the sign information is always incorporated



into the string; this is enforced so that the reduced string is always guaranteed to be, worst case, of equal length to the original string, and is highly likely to be shorter. Thus no (slow) memory allocations are required in the reduction algorithm, and also no memory leaks are possible[71].

A technique to take two such strings and combine them in such a way that the metric signature and the non-commutative rules of the algebra are utilised will now be presented. From Chapter 2 the anti-commutating behaviour of the algebra states that reversing the order of multiplication of two adjacent basis elements incurs an overall change of sign. Similarly, two adjacent elements, when identical, equate to plus or minus unity. Therefore, in order for reduction to be accomplished successfully, an arrangement method ensuring that all like elements are adjacent is required before the cancelling procedure is executed.

The arrangement method chosen here is a modified bubble-sort[44]. This mechanism functions as a standard bubble-sort except that the overall sign of the object is inverted in the event of an exchange of adjacent sub-elements. The bubble sort algorithm is chosen as the preferred arrangement mechanism over and above other (faster) sorting algorithms[75] because it always exchanges adjacent elements.

Although there are other sort algorithms which would be able of sorting this list with fewer operations; they do not facilitate the tracking of the sign changes in the algebra with sufficient reliability. This is essentially due to the fact that they do not tend to exchange adjacent sub-elements. Therefore the modified bubble-sort algorithm is the only guaranteed standard method for sorting the sub-elements whilst still obeying the commutation rules of Clifford algebras.

Once the sub-elements are arranged appropriately, the list is parsed by a second algorithm which removes adjacent like sub-elements and adjusts the sign appropriately based on which sub-elements were removed. This part of the algorithm is easily modified permit any conceivable metric. Finally the resultant sign is inserted at the beginning of the string, according to the representation convention, and the data returned. There are, however, two special cases which are trapped and modified before the string is returned. These special cases are those of the reversed cyclic permutations of (1, 2 and 3) the basis elements; i.e.  $e_{31}$  and  $e_{031}$ , both of which would be represented as  $-e_{13}$  and  $-e_{013}$  respectively unless taken care of (See Section 2.4, "Basis Elements"). Although this makes no

essential difference to the actual result, this convention enhances the readability of the elements.

An added feature of this algorithm, is that there is no requirement for the sub-elements to be numerical symbols. The algorithm performs equally well on letters or symbols. This algorithm is wrapped (Within Python) with a container (Grade) class which makes use of the algorithm as and whenever a new class is created.

### 3.3.6 Discussion & Summary

In this section the specifications for the algebraic software have been presented along with basic design concepts. Similarly, the programming language choice has been given with accompanying arguments for the final choice. Python has been shown to be the simplest and most effective means of meeting the specification and therefore is the programming language of choice here.

The object oriented approach has been shown to be a simple and effective means of implementing a symbolic algebraic system; also providing for simple debugging. The extensibility of this system is shown in Section 3.4.5, “Modules - Extending the Core System”.

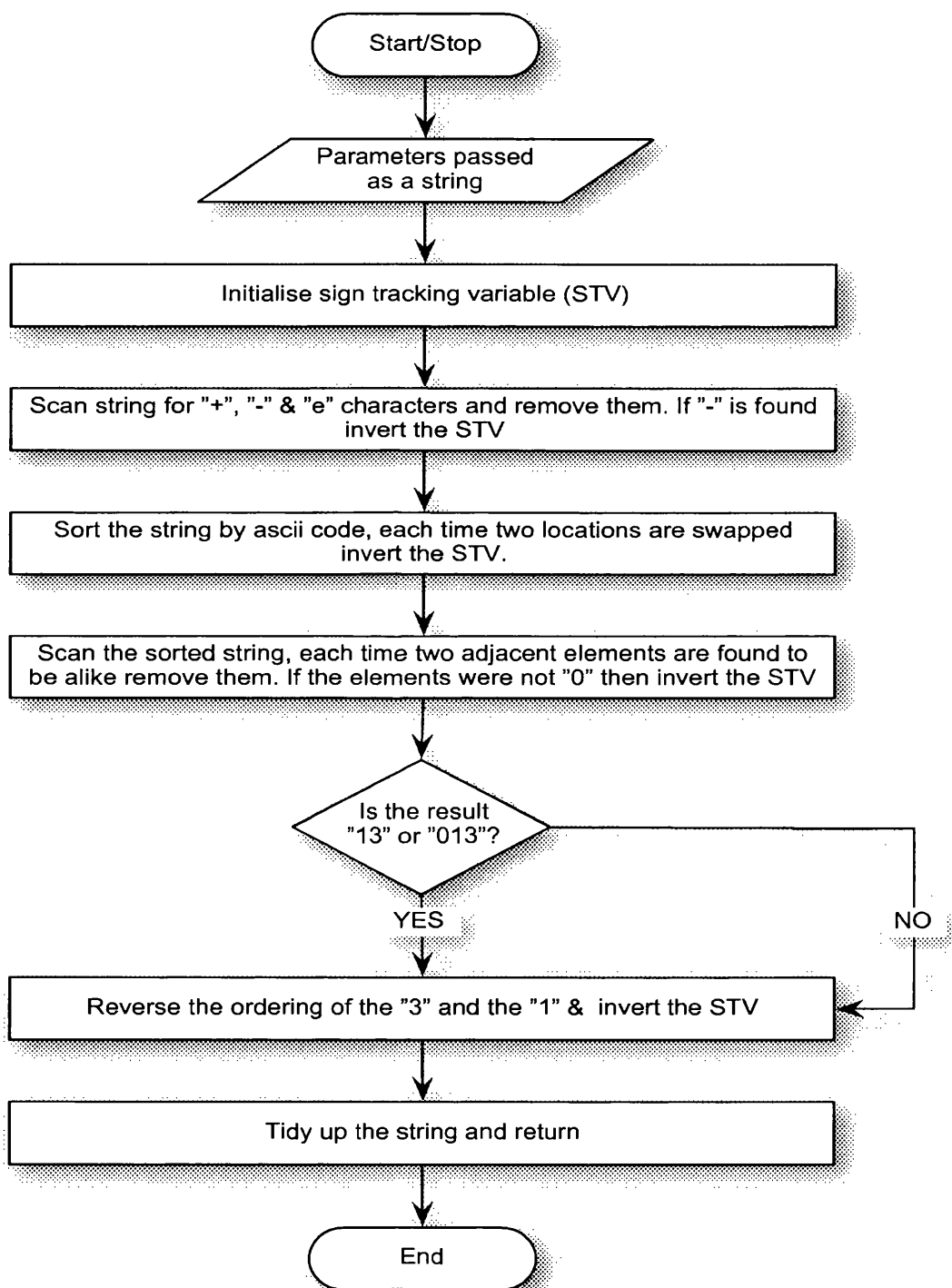
Finally, algorithm used for the reduction of spacetime basis elements has been introduced. This has been shown to be both robust and adaptable to any conceivable metric. This reliability is derived from the chosen sorting mechanism for the ordering terms prior to cancellation. The process of cancellation may be adjusted for different spatial signatures, thereby allowing the elements to individually square to plus or minus one. A system which yields the Lorentz metric (+ - - -) has been implemented here.

## 3.4 Methods, Techniques & Usage

### 3.4.1 Introduction

In this section some of the techniques and methods used to implement features and solve programming issues will be examined. This will involve a treatment of how numerics, variables and custom functions have been addressed in the program

### Spacetime Algebraic Reduction Algorithm



**Figure 3.1.** Flow chart showing the processes of the algebraic reduction algorithm. STV is the sign tracking variable.

design. Some special case problems will be treated and solved here using the facilities of Python (See Section 3.3.3, “Programming Language Choice”). For a basic guide to using the software see Appendix C.

The methods utilised for differentiation; and how this relates to the algebra discussed in Section 2.6, “The Spacetime Algebra” will be discussed. As mentioned in the specification (Section 3.3.2), integration will not implemented by this software model and is not discussed here.

How the core system of objects (See Section 3.3.4, “Object Oriented Approach”) may be extended to allow different types of functions will be examined here. This flexibility enables potential future developers to to extend the system to suit needs at a later date. No details of how the fine details of editing the code will be given here.

An introduction to wave functions and scripting in the software will also be presented in this section. This introduction will invoke some examples of how to manipulate basic wave function expressions using the software, and how these wave functions are expanded. Any limitations of such functions will also be discussed.

The introduction of the scripting system will demonstrate how one can use basic Python constructs to examine the phase space of a problem.

### 3.4.2 Numerical Evaluation & Variables

The algebraic software supports the introduction of a specific class of object called a “variable”. These variable objects differ from Python variables in that a Python variable is an instance of a particular class (for example a floating point number), whereas these algebraic variable objects are, by design, not assigned any particular value. The algebraic variable objects are designed to act as markers in expression and facilitate numerics and differentiation ( See Section 3.4.3, “Differentiation & The Differential Operator”). See Appendix C for how to create an instance of a `Variable()` class.

The numerical evaluation systems provided in the software permit one to represent, transform and numerically analyse hypercomplex multidimensional functions in the spacetime algebra. As a result functions with many dimensions can be visualised by using numerical sampling of expressions in this way.

In order for the numerical evaluation facilities to be able to function and perform the operations required, not only must it be able to substitute variables for numerical values, but it must also be able to separate out the different dimensions and return them as an “extracted” expression. These operations are provided in all the objects as methods, callable as “.eval()” and .seval().

There are two method hooks provided to allow numerical evaluation and the separation of the individual dimensions. These methods are named “eval” and “seval” respectively. The purpose of the “eval” function is to evaluate an expression which contains variables by replacing them with numeric values before returning a result. This is used most when plotting/graphing an expression. The purpose of the “seval” function (“single-dimension - evaluation”) performs the same action except it will return a single dimension of a numerically evaluated expression. This is used when one wants to examine/graph the behaviour of a particular dimension (i.e see what  $e_1$  is doing).

The values which are to be substituted for variables are passed to expressions in Python dictionaries. When an evaluation is being executed, each variable is called, where this dictionary is passed as a parameter. If the dictionary contains an entry for that variable, then the entry is returned, otherwise the object returns a reference to itself (in essence it returns itself, but by returning a reference less memory is used and therefore the system is faster). Using this system, an evaluation could conceivably replace a variable with an expression, but it is normal simply to provide a numerical value. See Appendix C for an example of “.eval()”.

Note that not all parameters need to be used, and that parameters may be used more than once during an evaluation cycle. In the case where a parameter is not provided for an expression, no error occurs; the variable will remain where it appeared before.

Similarly for the separation system, a call is made on the container object. However, in this instance, the parameter provided by the user may be any instance of any Python object. The separation evaluation system searches each *term* for an instance of the object passed. If a term contains such an object it will form part of the returned expression, otherwise it will not. For example, if one passes the object  $e_0$  to an expression through the use of the seval() method, then only those terms which contain an instance of an  $e_0$  object will be returned. This

method will not return terms which may originally have contained an  $e_0$  object, but which multiplied into another basis element to compose a higher or lower grade basis element. See Appendix C for an example of “.seval()”.

### 3.4.3 Differentiation & The Differential Operator

Differentiation of objects is implemented by method hooks[72]. By implementing a method (`__D__(x)`) for each type of object which allows it to differentiate itself with respect to a parameter variable, a general system for differentiation is generated. Each of these methods will return a new value, instance, term or expression which corresponds to the differentiated result. If an object cannot be differentiated with respect to the passed parameter then zero is returned. Higher classes, such as terms and expressions have the quotient and chain rules for differentiation are embedded into the methods described above.

One of the unique features of this implementation is that once all objects have differential hooks, Lambda operators can be utilised to define the overall differential operator required. In the case for the algebra defined in Section 2.6, “The Spacetime Algebra” the differential operator is: -

$$d = \partial_0 e_0 - \partial_1 e_1 - \partial_2 e_2 - \partial_3 e_3 \quad (1)$$

The lambda function is implemented in Python as[70, 73]: -

$$\text{lambda arguments : expression} \quad (2)$$

The differential operator is defined in the code as a summation lambda operation: -

```
D = lambda A : A.__D__(t)*e0
    - A.__D__(x)*e1
    - A.__D__(y)*e2
    - A.__D__(z)*e3
```

The result of a differential is a term or expression class instance. Extra differential operators which act as “proper” differentials in different inertial frames may also be defined by the same technique (See Section 5.3, “Proper Quantities & The Covariant Derivative”). The operator “Dprime” is created by: -

```
Dprime = lambda A : A._D_(t)*e0 *(gamma*e0+beta*gamma*e1)
        - A._D_(x)*e1 *(gamma*e1+beta*gamma*e1)
        - A._D_(y)*e2
        - A._D_(z)*e3
```

These new operators are then utilised in exactly the same way as the original differential operator (See Appendix C). A simple example of this is the differentiation of  $x^2e_{23}$ .

```
D(x*x*e23)
(-2.0*x*e123 )
```

And a slightly more complicated example of the differentiation of  $x\cos(x)e_{23}$ .

```
D(x*Cosine(x)*e23)
(-1.0*Cos[(x)]*e123+1.0*x*Sin[( x )]*e123)
```

which shows that the signs are tracked correctly.

### 3.4.4 Special Cases

There are some cases, where an attempt to multiply two objects of certain types can be handled by one of the two object's multiplication method, but not by the other. A typical example of this is a numerical object and a Grade class. In the case where the object which is able to handle this process (i.e the Grade class for example) is the first in the multiplication sequence, there is no problem as that object is called with the second object as a parameter. However, if the second object (e.g a numerical value) is multiplied into a Grade object, then an error will occur unless special care is taken from the programming point of view (e.g. because a numerical value is a Python "built-in" type, it has no information about Grade classes).

In order to overcome this problem, a "coercion" method is used which is called with the two objects as a parameter[73], in the which detects the types of the two objects and forms a Term() (Appendix C) instance based on the two parameters passed. The attempt to multiply two incompatible objects has resulted in the

operation being passed up the object hierarchy, so that a suitable container can be found. Similar procedures are utilised to ensure that objects which could not, by themselves handle the addition of themselves with a class instance are implemented. Most of the cases for which these procedure apply involve “built-in” classes within Python.

### 3.4.5 Modules - Extending the Core System

The core system of this software is extremely extensible and modified with relative ease. Any kind of commuting or non-commuting variable, basis element or operator may be implemented using this modular system. This is achieved by defining, in programming terms, how a new object interacts with objects of the same type. This is done, as explained earlier, through overloading the multiplication and division operations[72, 73].

This system of permitting extensible objects of different types and the ability to mix them all in same expression has allowed the implementation of wave functions in algebraic object form.

### 3.4.6 Scripting

The ability to embed the algebra into a real programming language and utilise this to execute complex or repetitive operations is one of the key features of the algebraic software presented in this chapter. This functionality is referred to as scripting. The ability to script such operations allows complicated and repetitive operations to be processed reliably (An example of this is whether a wave function is a solution to a wave equation).

As an example of how scripting is utilised, a particular script developed to produce real results can be found in Appendix C. This script was used to produce the tables Section 2.11, “Roots of Basis Elements”. This script utilises the programming features of Python, and combines them with the algebraic software module to produce a practical and tangible result. This program is two nested program loops which multiplies algebraic objects, in sequence, from an indexed list. The program then places the results in a table, then checks to see if the results are equivalent and if so, indicates as such on the output table. The output is then saved to disk using Python’s file I/O.



This type of scripting method was also utilised in the development and implementation of the “Root Search Algorithm” detailed in Section 3.5.3 and in the exhaustive search of Lorentz Transformations (rotations and boosts) presented in Section 5.2. Embedding this algebra into a programming language has enabled the development of new algorithms which process and compute problems in the spacetime algebra.

### 3.4.7 Discussion & Summary

The ability to script large phase space searches allows us to demonstrate and prove general and specific case mathematical rules within the algebra, by showing and cumulating all the possible outcomes for a particular operation or transformation. An example of this can be found in Chapter 2, Section 2.11 “Roots of Basis Elements”, where an exhaustive search to investigate the roots of basis elements was utilised.

## 3.5 Applications

### 3.5.1 Introduction

In this section some of the applications of the algebraic software developed in this chapter will be presented and discussed. These discussions will include short code examples and any performance issues of the software.

Wave function implementations will be introduced in this section. This will be accompanied by examples of the derivatives of wave functions. How the software is utilised to search for unique roots of basis elements will also be shown. The algorithm for finding unique roots follows from the work in Section 2.11, “Roots of Basis Elements”.

### 3.5.2 Wave Functions

In this section how wave functions are implemented and manipulated in the software will be examined. The wave functions will be dealt with primarily in exponential form and how the software permit the kinds of operations required discussed. For an overview of wave functions in the algebra presented in Section

2.6, see Section 4.6. The wave functions presented here will in some cases follow the form of those in Section 4.6. However the notation and utility of such functions will not be discussed at this point.

An example of a standard complex wave function is: -

$$A = A_0 \exp [i(kz - \omega t)] \quad (3)$$

This wave function is representable in the software as (Appendix C): -

```
A = A0*Wave( i*(k*z-w*t) )
```

The Wave() class constructor creates an instance of an exponential wave function, with the supplied arguments. In Section 4.6, “Wave Functions” wave functions using elements of the algebra in place of the unit imaginary are developed. An example of such a wave function is: -

$$\phi = \exp [ e_{12}(\omega t - k \cdot x_k)] e_{01}(1 + e_{03}) \quad (4)$$

which may be represented in the algebraic software as: -

```
k = 1
w = 1
z = Variable( 'z' )
t = Variable( 't' )
Phi = Wave( e12*(k*z-w*t) ) * e01*(1+e03)
```

The Wave() class has been developed with built-in intelligence, which it utilises to determine which power series (See Appendix B) to utilise when expressing a wave function in terms of composite elements. When asked to return the expression in expanded form, the function tests the non-scalar elements in each term contained within it’s arguments. This test determines whether or not the object squares to the positive or negative scalar. For the case where the overall squared sign of a term is positive (aggregated over all objects in the term), the hyperbolic sinusoidal functions are utilised in the expansion, otherwise normal sinusoids are used. Note that this test is done every time any wave class is expanded, and provides reliable results despite being a slow mechanism.

In addition to the standard set of operations (i.e. addition, subtraction and

multiplication), differentiation of wave functions is also supported. These features function by calling (as appropriate) the differential methods of the objects in the arguments. The `Wave()` class is able through the explicit use of `Term()` and `Expression()` install constructors (See Appendix B), to create and return a differentiated object as a new expression or term (as required).

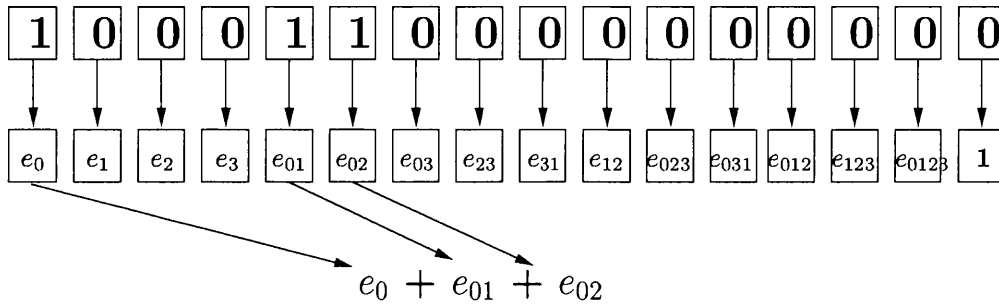
### 3.5.3 Root Search Algorithm

This next section will detail the software algorithm devised to search for roots of basis elements as introduced in Section 2.11, “Roots of Basis Elements”. The algorithm presented here is designed to be a system capable of reliably scanning the entire phase space of all combinations of the basis elements; and also all combinations of signs over all possible sets of these elements; without any repetition; effectively finding the roots by “brute force”.

An iterative state machine is utilised here in preference over recursive system[76] as the latter has the potential for repetition unless great care is taken. The iterative method is chosen to index a simple fixed list of the basis elements. This fixed method of indexing basis elements in a numerical fashion produces unique sequenced combinations of element sets. A binary counting mechanism is then used to reference this list mechanically (See Figure 3.2). As the binary counter counts, those bits which are set are utilised to “pick” the elements out of the list. As the nature of a binary counter is such that all possible combinations are exhausted, the desired result of exhausting each and every possible combination with no overlap has been achieved.

A similar procedure is implemented for exhausting all the possible sign combinations for each and every element combination. An  $N$  digit binary counter is generated on-the-fly for each element combination, where  $N$  corresponds to the number of terms in the test expression. In this case, however, a set digit represents a positive sign for a term and a clear bit represents a negative term. The relative positions of the bits make a direct correspondence to the trial expression.

Through the use of the object oriented system, described in Section 3.3.4, “Object Oriented Approach” the program is able to simply square, or cube the trial expressions and examine the object returned. Due to the fact that the terms and expressions are formed from Python sequence objects[73], determining the



**Figure 3.2.** This diagram shows how a 16 bit binary counter is utilised to form expressions from the 16 basis elements.

number of terms in an expression is a trivial operation\*.

The ability to ascertain this information enables the software to determine if the trial expression and when squared, yields a term containing a single basis element. This method, in effect, determines all the possibilities and implements a check procedure that can be executed by a computer program. The  $\frac{1}{\sqrt{N}}$  factor in Eq. (111) & (112) is the standard normalisation step and is based on the number of terms in the trial solution (N) and the power of the trial root under examination (K).

In addition, a bit count filter which can reject all numbers in the binary count sequence which do not have a specified number of bits set is used when generating the tables presented in Section 2.11, “Roots of Basis Elements”.

### 3.5.4 Discussion & Summary

In this section an algorithm utilised to exhaustively search the phase space of algebraic combinations of roots of basis elements has been introduced. Through the use of auto-generating binary counter methods, all the possible expressions with a particular number of basis element terms can be generated. These are then tested to ascertain whether or not they are the roots of any elements. The method is a simple “brute force” mechanism. The results produced by this algorithm can be found in Section 2.11, “Roots of Basis Elements”.

\*In this case `len(<expression instance>.expression)` returns the desired information.

Wave functions and the `Wave()` class have been introduced in this section. How a `Wave()` class may be utilised to represent standard complex wave functions as well as wave functions using elements of the Clifford Algebra  $\mathcal{Cl}_{1,3}$  has been demonstrated. Mechanisms for appropriately expanding the `Wave()` class in terms of appropriate power series functions have also been introduced.

## 3.6 Discussion

The algebraic software presented in this chapter has been developed in accordance to the specifications given in Section 3.3.2, “Specification”. The requirement for the development of algebraic software stems from the lack of suitable products available to perform the kinds of operations required in the spacetime algebra (Section 2.6, “The Spacetime Algebra”). Although there are a number of packages available to manipulate geometric and Clifford algebras, unfortunately none of the available packages are suitable. Most of these packages do not support the metric chosen for investigation here. Also the lack of any support for the vector differential operator has also been a crucial factor in justifying the development of this software; as without this one cannot determine whether or not a wave function is a solution to a wave equation.

During the course of this implementation process several decisions and paths have been adopted where a different choice may have been valid or perhaps in some instances produced a more user friendly system, faster code execution, lower memory usage or improve some other aspect of the system which may be desirable. However, the software has always been developed with the needs of the research in mind, rather than the software dictating the direction of the research. Functions have been added to the code as required for the purposes of research, rather than spending lengthy time periods developing needless features. These abilities have been developed over a long period to permit the execution of operations critical to the research presented in this thesis.

An example of a possible improvement may be to convert the system to utilise the C/C++ programming language, rather than the current Python/C combination. Obviously this would increase the performance of the code (by at least 10%), as C/C++ are compiled languages. However, as discussed in Section 3.3.3, to do so would increase the time taken to develop new features, increase the complexity

and readability (for the less experienced developer), would increase the overheads and lead times required to develop a functional system. Indeed, as discussed in previous sections, whilst performance is an issue, one can argue that the benefits of utilising the Python programming language (in this instance) outweigh the disadvantages experienced in overall performance.

The benefits of the scripting system has also become an advantage brought about by the usage of the Python programming language. By utilising Python, a fully tested scripting system is available to the user for no further development effort. The fact that the Python programming language has undergone years of testing and debugging has enabled the development to focus on those parts of the software concerned with algebraic manipulations and other features required for research. Had the complete development of a scripting system been required, then many months of testing and debugging would have been required.

Similarly, the mechanisms employed in the algorithms developed may not always be the primary choice of those wishing to maximise performance of the code. In particular, the bubble-sort[44, 45] method is known to be one of the slowest mechanisms for sorting and does not take advantage of any initial partial ordering[77]. However, as discussed in Section 3.3.5, “Basis Element Reduction” the reduction mechanism is able to track signs properly in the context of the mathematics by ensuring that only adjacent entries in the list are exchanged. Faster algorithms such as a quicksort[75], whilst faster in execution, would not allow the tracking of signs properly and therefore not be able to produce reliable results. The exhaustive algorithm utilised to find roots of multivector quantities is another, less prominent example of where speed has been compromised for reliability of the results it produces. Here a system which produces all possible expressions, with coefficients of equal magnitude, containing terms of a specified length (2 terms, 3 terms etc) is utilised and simple tests are performed to see if this term is a root of a basis element. Whilst slow, this mechanism is guaranteed to find all the roots, with coefficients of equal magnitude, in a finite amount of time.

In this chapter some of the methods and techniques used to implement the algebraic software have been presented and used to show some of the uses to which this software can be applied. How scripting can be used to obtain tangible results

has been illustrated. Solutions to special case problems with Python's "built-in" types have also been given along with numerical evaluation and separation techniques.

The utility of scripting in performing complex operations in both a repeatable fashion and also in producing tabulated results has been shown to be invaluable. How this feature has enabled the behaviour of the three dimensional algebra  $\mathcal{Cl}_{3,0}$  to be determined and compared to the four dimensional algebra  $\mathcal{Cl}_{1,3}$  has been shown in this section. In addition, the software presented here has been used to produced the results presented in Sections 2.11, 5.2 and 5.4. The scripting power of the algebraic software has played a key role in ascertaining these results. It is likely that these results would have taken much longer to obtain without the software.

Whilst in some respects this software has a great deal of scope for improvement, the fact that the facilities clearly exist to improve and documentation exists to assist the implementation of this is seen as a benefit rather than a disadvantage. Obviously the software does not perform all the operations one might conceive, but the facilities do exist to allow these operations to be incorporated into the software at a later date, with less effort than would otherwise be required. The software has been presented as a functioning framework, capable of assisting the researcher, rather than a rigid and inflexible system which forces the user to limit or compromise the types of operations that may wish to performed.

### 3.7 Chapter Summary

In this chapter the development of the algebraic reduction software has been justified and presented. Some of the various uses to which this software may be applied have been demonstrated. It has been shown that there are a number of packages available to manipulate geometric and Clifford algebras; unfortunately, at the time development commenced, none of the available packages were suitable for research purposes. The lack, in most cases, of support for different metrics and a scripting facility were the greatest factors in the justification for development of a custom package. The software developed here is, at the time of writing, still the only package capable of properly supporting differentiation in  $\mathcal{Cl}_{1,3}$ .

The software is able to perform algebraic reduction of basis elements, as required in the specification. The software utilises the spacetime reduction algorithm (See Section 3.3.5, “Basis Element Reduction”) to efficiently reduce basis elements to their most simple form. The Lorentz metric (+ - - -), anti-Lorentz metric (- + + +) and Euclidean metric (+ + +) have been implemented. In order to prevent confusion, basis elements in the Lorentz metric (+ - - -) are always represented by the letter “e”, whereas the other two are both represented with the letter “h”. The alternate metrics have been implemented to allow comparison with results obtained in the Lorentz metric (+ - - -) (See Section 2.6). As required by the specification, steps have been taken to ensure that the order of multiplication is preserved at all times.

Scripting of algebraic operations has been implemented as per the specification. The scripting system allows a number of operations to be performed as one and then re-run at will. Providing repeatability in results. This scripting system not only allows the execution of algebraic type operations, but because Python has been chosen as the development language, the user is also equipped with a full suite of programming libraries and functions. The software presented here has, in effect, enhanced Python with spacetime algebra capabilities. This system allows an individual with little programming experience to write spacetime algebra programs. The scripting features have been utilised in the investigation of square roots of the basis elements presented in Section 2.11, “Roots of Basis Elements”.

Additionally, this software is capable of numerical evaluation and dimensional separation of terms. This latter feature has been shown to be essential if one particular quantity is to be isolated and visualised. The core of the system has been shown to be extensible and where necessary portions of code which are heavily utilised can be optimised and compiled into C/C++.

The algebraic software presented in this chapter has fulfilled the specifications and although other software now exists which is capable of performing many of the required operations, they were not available when development commenced and still do not have all the required features.



# Chapter 4

## Field Equations

### 4.1 Introduction

The aim of this chapter is to develop the familiar electromagnetic field equations using the spacetime algebra presented in Chapter 2. The utility of this work is manifold; not only to provide a basis for comparison with work done by others, but also to enable the investigation of the relativistic properties and symmetries of the algebra; thus providing insight into the behaviour of new kinds of wave functions offered by the algebra.

Maxwell's equations will be developed in this chapter using the algebra presented in Chapter 2. The aim being to present the derivation of the field equations and show that the complete set of these equations appear in a particularly compact and elegant form ( $dF = J$ ). This derivation will begin with discussion of the possible starting points and the relative merits and demerits of each. Later in this chapter there will be a comparative analysis of this work with the work done by others. In particular, the work by Gull *et al*[17], Morse *et al*[78] and Jackson[30] will be used as a basis for this.

This chapter is mainly a review of electromagnetism in Clifford algebras. The exceptions to this are Section 4.4 which is a re-examination and Section 4.6.5, "New Wave Functions" which is original work.

Once the field equations are introduced and analysed, solutions to these equations will be derived. These solutions will take the form of travelling waves using elements of the algebra instead of the unit imaginary. The various polarisations of

these solutions will be investigated; linear superposition along with normalisation steps will also be demonstrated. There will be a section on a set of alternative wave functions within this algebra which at first glance do not appear to be conventional travelling wave functions; but nevertheless are valid propagating solutions to the field equations presented.

The behaviour of these wave functions under Lorentz transformation will be examined in detail in the following chapter (See Section 5.2, “The Lorentz Transformation”).

The gauge degree of freedom is introduced in this chapter through the scalar. This degree of freedom is used to show how setting the various portions which contribute to the scalar simplifies the dynamics of field equations. Several simplifications will be introduced a correspondence made with well known gauge conditions. Finally, the differences between gauge conditions and the gauge itself will be discussed.

## 4.2 Potentials & Fields

Throughout this chapter fields and their dynamics will be examined mathematically. This will commence by examining and discussing how an expression for, and the form of, the fields themselves are derived. This is done before examining the dynamics (See Section 4.3, “The Maxwell Equations”). The electric and magnetic fields can be found in terms of the derivative of a potential. There are several possible choices for the nature of this potential; e.g. a scalar plus three-vector or a full four dimensional vector. It is also possible to choose the fields as the starting point rather than the potentials. However, in so doing, a choice must be made for the form of each field rather than allowing the mathematics to be the guide. All of these quantities (the potentials and the fields) are related to each other and all are known to be significant in the role they play in electromagnetism and in the coupling with quantum mechanics[79]. However, the final justification for any particular choice is found in the utility of that approach and the way in which the results fit with the known facts.

A four-potential is chosen here as the basis for the development. The reason for this is manifold, firstly, in making this choice, one is guided to the form (grade) of the fields when the derivative is taken. Secondly, the work of Aharonov

& Bohm[79] proposed that the absolute value of the potentials, and not simply the temporal and spatial rates of change of the potentials (i.e. that of the fields amongst others), have a significant effect on the outcome of an event in quantum mechanics through the minimal coupling.

The four-potential is defined as: -

$$A = e_\mu A_\mu = e_0 A_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} \quad (\text{Summation}) \quad (1)$$

The four derivative (see Section 2.6.7, “Differentials”) of this quantity is defined here as: -

$$dA = d \cdot A + d \wedge A \quad (2)$$

and yields a scalar (L) plus six bivector components (F). The six bivector components can be split into two sets of three, space-like ( $e_{ij}$ ) and spacetime-like ( $e_{0i}$ ) (See Section 2.3, “The Clifford Product of Vectors”); of which the space-like correspond to the magnetic field and the spacetime-like correspond to the electric field. The unfamiliar form of the electric field (bivector) will be discussed later. The magnetic field tensor is denoted as  $\vec{B}$ , the electric field tensor as  $\vec{E}$  and the remaining scalar component as  $L$ . More explicitly the 16 terms of the product  $dA$  may be written as: -

$$dA = \partial_0 A_0 + \nabla \cdot \vec{A} + \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} (\partial_0 \vec{A} + \nabla A_0) - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \nabla \times \vec{A} \quad (3)$$

which can be separated as: -

$$d \cdot A = L \quad (4)$$

$$d_0 e_0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A}_i - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \nabla e_0 A_0 = \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} \vec{E} \quad (5)$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \nabla \times \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} = \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \vec{B} \quad (6)$$

Extra degrees of freedom can be introduced through the scalar  $L$ . Gauge conditions, for example, are introduced in this way and the Lorentz gauge here corresponds to  $L = 0$  ( $d_\mu \cdot A_\mu = 0$ ). Other gauges such as the Coulomb gauge ( $\nabla \cdot \vec{A}$ ) are introduced by an appropriate choice elements which contribute to this scalar. This will be explained in more detail later in Section 4.4, “The Gauge”.

To aid visualisation the familiar three-space forms are used. These include such quantities as  $\vec{E}$ ,  $\vec{B}$  and  $\vec{J}$  (electric field, magnetic field and current density respectively), and the standard dot and cross product. The full four-space algebra is maintained by means of a positional column notation for the components. As a result of this convention, the rules of the full four-space algebra are respected whilst still permitting a mathematical representation in the familiar form of vector calculus. This not only allows expanded equations to be written in a compact form, which is useful as the number of terms becomes large in more complex equations, but also greatly assists physical intuition. The field is written as: -

$$F = - \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} \vec{E} - \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \vec{B} \quad (7)$$

In Eq. (7) the electric and magnetic fields take a bivector form  $e_{0i}$  and  $e_{ij}$  respectively, and appear neither as polar or axial vectors nor as a set of tensor components. Explicitly, the familiar three-space polar vectors  $\vec{J}$  and  $\vec{E}$ , and the three-space axial vector  $\vec{B}$  are projections of the four-current  $J$  and field  $F$  on Cartesian three dimensional space using the following transformation rules: -

$$e_i \rightarrow \hat{x}_i \quad (\text{polar vector}) \quad (8)$$

$$e_{0i} \rightarrow \hat{x}_i \quad (\text{polar vector}) \quad (9)$$

$$e_{jk} \rightarrow \hat{x}_i \quad (\text{axial vector}) \quad (10)$$

In particular, the  $\vec{E}$  field is treated as a space-time bivector and not as a vector. This is simply because it follows from the definition of the four-potential and the derivative.

## 4.3 The Maxwell Equations

### 4.3.1 Introduction

In this section the dynamics of the fields (See Section 4.2, “Potentials & Fields”) will be examined. At the level of the field dynamics the utility of the algebra presented in Chapter 2 becomes apparent.

The investigation into dynamics begins by introducing the derivative of the fields in the absence of any charges. The equations that appear are the full set of source free Maxwell’s equations. This will be followed by the introduction of the inhomogeneous Maxwell’s equations, which corresponds to an electromagnetic system with a non zero four-vector current source term  $J$ . Finally this derivation and these methods will be compared with electromagnetism in relativistic notation[30].

### 4.3.2 Source Free Equations

The field equations, in the absence of sources, can be written in terms of the dynamics of the four-vector potential  $A$  as follows: -

$$d(dA) = dF = 0 \tag{11}$$

This equation represents electromagnetic systems in the absence of charges or sources. For this reason this form of Maxwell’s equations are also known as the source-free equations. Note that the expression  $d(dA)$  is not entirely equivalent to the second order derivative  $d^2A$  (See Section 2.6.7, “The Quotient”). Eq. (11) can be expanded in terms of the more familiar Maxwell’s equations as: -

$$e_0 \nabla \cdot \vec{E} = 0 \quad (12)$$

$$e_{123} \nabla \cdot \vec{B} = 0 \quad (13)$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (\nabla \times \vec{B}) = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (\partial_0 \vec{E}) \quad (14)$$

$$\begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (\nabla \times \vec{E}) = - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (\partial_0 \vec{B}) \quad (15)$$

The multivector units of each expression have been preserved here. Eq. (12), (13), (14) & (15) represent the homogeneous Maxwell's equations in the notation presented for the spacetime algebra in this thesis.

### 4.3.3 Inhomogeneous Equations

Consider the case where the dynamics of the field equations are not equated to zero. i.e.  $dF \neq 0$ . The conventional approach is to introduce a four-current term  $J$ , giving: -

$$d(dA) = d(F + L) = J \quad (16)$$

Setting the gauge  $L = 0$  everywhere, so that the condition  $dL = 0$  (See Section 4.4, "The Gauge") is imposed: -

$$dF = J \quad (17)$$

This equation now includes terms that correspond to the sources. The expansion of Eq. (17) gives the full set of Maxwell's equations: -

$$e_0 \nabla \cdot \vec{E} = e_0 J_0 \quad (18)$$

$$e_{123} \nabla \cdot \vec{B} = e_{123} J_0^m = 0 \quad (19)$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (\nabla \times \vec{B} - \partial_0 \vec{E}) = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{J} \quad (20)$$

$$- \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (\nabla \times \vec{E} + \partial_0 \vec{B}) = \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} \vec{J}^m = 0 \quad (21)$$

Eq. (18), (19), (20) & (21) are the full set of Maxwell's equations, with sources, in the algebra presented in this thesis (Chapter 2). Note that the introduction of the four-current term  $J$  has led naturally to the introduction of the magnetic monopole four-current term  $J^m$ . Although this term is carried in the equations, it will be set to zero at all times. Working out the full geometrical product for Eq. (17),  $dF = d \cdot F + d \wedge F$ , vector and trivector parts are identified. These correspond to the inhomogeneous and homogeneous Maxwell's equations respectively:

$$d \cdot F = J \quad (22)$$

$$d \wedge F = 0 \quad (23)$$

The trivector part Eq. (23), as discussed above, has no source term, expressing the absence of magnetically charged monopoles, consistent with a vector potential description where  $\vec{B} = \nabla \times \vec{A}$ .

#### 4.3.4 Other Formalisms

The development of the field equations in a different formalism will be examined here. The usual relativistic notation[30] is utilised to express the dynamics of the fields\*. The fields in this formalism are introduced as a field strength tensor as

---

\*The example is taken and adapted from J.D Jackson[30].

follows: -

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad (24)$$

Note that this field strength tensor may be derived in terms of a three potential and a scalar field  $(\Phi, \vec{A})$ . The covariant differential operator within this formalism[30] is given as a vector with the following properties: -

$$d_\mu = \left[ \frac{d}{\partial t} \quad \frac{d}{\partial x} \quad \frac{d}{\partial y} \quad \frac{d}{\partial z} \right] \quad (25)$$

The dynamics of these fields can be found using the same approach as used for spacetime algebra; i.e. the four-derivative of the fields. This is given by the product: -

$$d_\mu F^{\mu\nu} = \left[ \frac{d}{\partial t} \quad \frac{d}{\partial x} \quad \frac{d}{\partial y} \quad \frac{d}{\partial z} \right] \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} = J^\nu \quad (26)$$

which when fully expanded (See Appendix A) corresponds to: -

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \quad (27)$$

$$\nabla \times c\vec{B} - \frac{d\vec{E}}{dt} = \vec{J}/\epsilon_0 \quad (28)$$

This, however, is only half the Maxwell terms. In order to obtain the remaining terms one requires the introduction of the dual field strength tensor (See Section 4.5, "The Dual Field Strength Tensor").



## 4.4 The Gauge

### 4.4.1 Introduction

In this section the gauge will be introduced and discussed. The work presented here follows, in part, the work of Dirac, Jackson, Novak & Stevens[80, 30, 81, 82]. A demonstration of how algebra presented in Section 2.6, “The Spacetime Algebra” may be utilised to extend and simplify the field equations in terms of a homogeneous wave equation of the four potential, which still supports the introduction of charge density through the gauge will be presented here. This will begin with the introduction of general gauge transformations, before progressing to introduce the Lorentz and Coulomb gauge conditions.

### 4.4.2 Gauge Transformations

A gradient (four gradient) of an arbitrary scalar function  $\Lambda$  may always be added to the four potential without changing the fields (electric or magnetic): -

$$A \rightarrow A' = A + d\Lambda \quad (29)$$

For arbitrary  $\Lambda$ . This is an example of a *gauge transformation* of the system. The invariance of both sets of the fields under such a transformation is known as *gauge invariance*[30]. Note however, that the expression “gauge invariance” does not, of itself, imply the invariance of the fields under a gauge transformation; but rather refers to the invariance of certain quantities under a gauge transformation.

### 4.4.3 Gauge Conditions

The potentials  $A_\mu$  for an arbitrary field are not uniquely defined. It is possible, amongst other possibilities, to restrict the arbitrariness of the potentials  $A_\mu$  by setting the condition[81]: -

$$d_\mu \cdot A_\mu = 0 \quad \mu = \{0 \dots 4\} \quad (\text{Summation}) \quad (30)$$

which is referred to as the Lorentz gauge condition[30, 82, 81, 80]. Note this is the diagonal portion of this product only. Other authors[30, 82] may express this

condition in terms of the relationship between the temporal and spatial gradients of the four potential: -

$$L = \frac{\partial}{\partial t} A_0 + \nabla \cdot \vec{A} = 0 \quad (31)$$

Although expressed in a perhaps more familiar form[30], Eq. (31) is exactly the same relation as Eq. (30). All of these quantities are scalars. As noted earlier; within the formalism presented here, this condition can be defined by simply setting  $L = 0$  everywhere.

Another gauge of interest in electromagnetism is the Coulomb gauge[30, 82]. This gauge is also sometimes referred to as the radiation gauge or transverse gauge[30]. Just as for the case of the Lorentz gauge, the Coulomb gauge condition may be expressed in terms of the potential  $A$ , such that Eq. (32) below is the Coulomb gauge condition[30, 82].

$$\nabla \cdot \vec{A} = 0 \quad (32)$$

Imposing these gauge conditions (Lorentz and Coulomb) on the four potential, allows simplifications to be made to the field equations. The Maxwell's, as equations presented in Section 4.3, are an example of such a simplification. These simplifications are often of great assistance in obtaining solutions to the field equations[30]. From Eq. (3) in Section 4.2, "Potentials & Fields", The full dynamics of fields may be written down in terms of the potentials as: -

$$d(dA) = d\left(\frac{\partial \vec{A}}{\partial t} - \nabla \times \vec{A} + \nabla A_0\right) + d\left(\frac{\partial A_0}{\partial t} + \nabla \cdot \vec{A}\right) = dF + dL \quad (33)$$

It is clear that setting the Lorentz gauge described earlier ( $L = 0$  everywhere for all time), vastly simplifies the dynamics of the fields, as the  $\frac{\partial A_0}{\partial t} + \nabla \cdot \vec{A}$  part of the expression vanishes. This leaves only three of the original five terms. Similarly, incorporating the Coulomb gauge allows the  $\nabla \cdot \vec{A}$  portion of Eq. (33) to disappear, simplifying the  $dL$  portion of the equations to: -

$$dL = d\left(\frac{\partial A_0}{\partial t}\right) \quad (34)$$

Note that there may be a difference between imposing a gauge condition and setting the gauge; although in some situations the two are in fact the same. A

gauge condition, in the context of electromagnetism, is a restriction or function assigned to a portion of the four-potential which simplifies the equations so that they can be solved. On the other hand, the gauge is only *set* once every degree of freedom has been exhausted. An example of a gauge condition that also sets the gauge is the aforementioned Lorentz gauge. The gauge is set because there are no remaining free parameters in the potentials. Conversely, the Coulomb gauge condition does not completely set the gauge. By setting only  $\nabla \cdot \vec{A} = 0$ , there remains a free parameter in the form of the time derivative of  $A_0$ , therefore not every gauge degree of freedom has been set.

There are many other gauge conditions through which simplifications can be made. A speculative gauge condition will now be presented in the hope of reducing the four potential to a homogeneous wave equation. This gauge condition is:

$$d\left(\frac{\partial A_0}{\partial t} + \nabla \cdot \vec{A}\right) = dL = -J \quad (35)$$

where  $J$  is a four-vector current as discussed in Section 4.3.3. The motivation for this choice being that the dynamics of the fields now obey a homogeneous wave equation  $d^2(A) = d(dA) = 0$ . Remembering that  $dF = J$ : -

$$d^2 A = d(dA) = d(F + L) = dF + dL = 0 \quad (36)$$

Eq. 36 allows both current and charge density to be set through the gauge, whilst at the same time obeying the homogeneous wave equation  $d^2 A = 0$ . In addition, by defining  $d^2 A = 0$  and  $dL = -J$  the possibility exists to set the charge, through the introduction of a *scalar field* together with the gauge. In order to do this, the gauge cannot be global, but instead must change close to the position of the charge. Here the time derivative of the scalar L corresponds to a divergence of the electric field (i.e. a charge). This can be shown to be true[38] for the case where  $d^2 A = 0$  by: -

$$-\frac{\partial L}{\partial t} = -\frac{\partial^2 A_0}{\partial t^2} - \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) \quad (37)$$

$$= -\nabla^2 A_0 - \nabla \cdot \left( \frac{\partial \vec{A}}{\partial t} \right) \quad (38)$$

$$= -\nabla \cdot \left( \nabla A_0 + \frac{\partial \vec{A}}{\partial t} \right) \quad (39)$$

$$= \nabla \cdot \vec{E} \quad (40)$$

#### 4.4.4 Discussion

There are however some consequences which must be noted when deviating from a global gauge condition in this manner. Firstly, by introducing a charge density in this way, local charge conservation is lost. In addition, when this deviation is undertaken, one destroys gauge invariance.

These problems have been the subject of discussion in the literature. In particular, Novak[81] and Dirac[80] discuss the possibility of introducing electron structure in an effort to avoid the infinite self-energy problem associated with point like particle theories, and introducing a more powerful electrodynamic theory in the process. Novak[81] does, however, suggest that the existence of gauge transformations “*signifies that there are more variables in the theory than necessary, from the physical point of view*” and that as a consequence that “*the lack of gauge invariance is not a significant drawback*”. Novak continues by suggesting that, instead of being a drawback, the existence of these degrees of freedom may be utilised in the introduction of charges; rather than “*bringing in new variables to describe them, as is done in the classical theory*”.

The investigation of the electron structure in this context is beyond the scope of this thesis. This approach may find utilities in a future study of these topics.

## 4.5 The Dual Field Strength Tensor

### 4.5.1 Introduction

The dual field strength tensor as presented by Jackson[30] will be introduced and examined in this section. Although the introduction of this dual is not required in the algebra presented in this thesis to give all the terms for the source free, homogeneous and inhomogeneous Maxwell's equations, it is perhaps of benefit to the interested reader to firstly investigate why this is the case, and secondly demonstrate a practical application of the duality operator itself. For a fuller discussion of the duality principle See Section 2.7, "The Duality Principle & Duality Operations".

From Jackson[30] Page 556, the dual field strength tensor  $F^{\alpha\beta}$  is a duality transformation of the fields,  $F^{\mu\nu}$  such that: -

$$\vec{E} \longrightarrow \vec{B} \quad (41)$$

$$\vec{B} \longrightarrow -\vec{E} \quad (42)$$

### 4.5.2 The Field Tensor

The field strength tensor under a (Hodge) duality transformation in relativistic notation is illustrated by[30]: -

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \longrightarrow F^{\alpha\beta} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{bmatrix} \quad (43)$$

In the previous section the following product was introduced: -

$$d_\mu F^{\mu\nu} = J^\nu \quad (44)$$

The properties of the dynamics of the field and dual field tensors  $F^{\mu\nu}$ ,  $F^{\alpha\beta}$  will now be examined. The dynamics of these tensors are defined by: -

$$d_\mu F^{\alpha\beta} = 0 \quad (45)$$

giving: -

$$\nabla \cdot \vec{B} = 0 \quad (46)$$

$$-\nabla \times \vec{E} - \frac{d\vec{B}}{dt} = 0 \quad (47)$$

Giving the pair of homogeneous Maxwell's equations. This duality operation has been introduced and demonstrated for the spacetime algebra in Section 2.7, "The Duality Principle & Duality Operations". There is, however, no requirement for a dual field strength tensor in the derivation of field equations.

### 4.5.3 Discussion

There is a direct correspondence between the terms that appear for the field and dual field strength tensors and those which arise from the inner and exterior products in the Clifford-Dirac algebra. The result from  $d_\mu F^{\mu\nu}$  appears to correspond very closely to the inner differential product, i.e.  $d \cdot F = \vec{J}$  (Eq. (22) ) whereas the dual of this  $d_\mu F^{\alpha\beta}$  corresponds closely to  $d \wedge F = 0$  (Eq. (23)). Therefore it is clear that in the case of the usual relativistic notation, the choice of algebra introduces the requirement for the dual field strength tensor.

## 4.6 Wave Functions

### 4.6.1 Introduction

This section is concerned with the development and presentation of wave equation solutions to the source free Maxwell's equations. One of the salient features of the algebra presented in this thesis is that it permits the development of a single wave function which contains both the electric and magnetic components of the fields; with purely real components. The wave functions presented here will not contain the imaginary number  $i$ . The presentation of wave function solutions and a study of their behaviour is paramount to the development of any field theory.

The development of a standard set of wave equation solutions will be presented in the following section. This will be followed by a study of the behaviour of those functions and will be accompanied by the introduction of polarisation states of the wave functions. Subsequently there will be an investigation into the behaviour of those states, in order to confirm that they are true to what is observed physically. Comparisons will then be drawn with wave functions and polarisation states from other formalisms[30] and relative advantages and disadvantages of each discussed. Superposition of the polarisation states will also be demonstrated. The primary intention here is to demonstrate that wave functions within this algebra can be expressed in purely real form.

In addition, a study of some unusual wave functions will be presented later in this section. These wave functions are curious in that upon first inspection one would not expect them to represent travelling waves. Showing that these unusual functions are in fact travelling wave solutions to the source free Maxwell's equations is a further aim of this section.

The properties of these unusual wave functions will then be examined in detail. The aim being to show that these wave functions, whilst still being solutions to the source free Maxwell's equations, are unique to the algebra presented here. A further aim is to show that these unusual wave functions do have some peculiar properties under certain types of reflections, which may or may not be physical. The potentially non-physical properties of these wave functions will be discussed later.

The discussion on the behaviour of these wave functions under Lorentz transformations will be carried out in Chapter 5, "Lorentz Transformations".

### 4.6.2 Standard Wave Functions

Transverse wave solutions to the Maxwell's equations presented in the previous section can be found by analogy with wave solutions in complex form[30]\*: -

$$\mathbf{E}(\mathbf{x}, t) = (\epsilon_1 E_1 + \epsilon_2 E_2) \exp [i\mathbf{k} \cdot \mathbf{x} - i\omega t] \quad (48)$$

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\*These equations are taken from "Classical Electrodynamics", 3<sup>rd</sup> Edition by J.D Jackson and can be found on page 299.

$$\mathbf{B}_j = \sqrt{\mu\epsilon} \frac{\mathbf{k} \times \mathbf{E}_j}{k}, \quad j = 1, 2 \quad (49)$$

where the amplitudes  $E_1$  and  $E_2$  are complex numbers and the field magnitude is given by: -

$$E = \sqrt{E_1^2 + E_2^2} \quad (50)$$

These wave functions contain complex components and require two separate equations for the electric and magnetic descriptions. Let  $\phi$  be a plane wave solution to the source free Maxwell's equations. In what follows spatial position variables are denoted  $x_i$ .

$$\phi = \exp[\mathbf{M}(kx_i - \omega t)] \phi_0 \quad (51)$$

In Eq. (48) & (49) the electric and magnetic components of the field are represented in separate equations. In this algebra there exists the possibility to represent the fields in a single equation  $\phi = \phi_{prop}\phi_0$  where  $\phi_0$  is a non-scalar sum of field elements which form a chirality operator:-

$$\phi_0 = e_{0j}(E_0 + B_0e_{0i}) \quad (52)$$

where  $E_0$  and  $B_0$  are scalars. Various polarisation states can be achieved through the juxtaposition of signs and elements in Eq. (52). This will be examined in more detail in Section 4.6.4, "Polarisation States of Standard Wave Functions". In order for Eq. (51) to support travelling waves,  $\mathbf{M}$  in Eq. (51) must be replaced by an element of the algebra which is a member of  $e_{\ominus}$ . The choice of  $e_{\ominus}$  will vary for a particular solution. In this algebra transverse waves, analogous to Eq. (48) & (49), can be found by choosing a  $e_{jk}$  element for  $\mathbf{M}$  in Eq. (51) such that it is linearly independent of the propagation direction  $\vec{S}$ . This effectively substitutes  $i$  for an element of the algebra which is isomorphic to the quaternion which matches the propagation direction of the wave[35, 55].

$$\phi = \exp[e_{jk}(k \cdot x_i - \omega t)]e_{0j}(E_0 + B_0e_{0i}) \quad (53)$$

Eq. (53) is a description of electric and magnetic fields rotating about the  $x_i$  axis; there is no rotation into or from a complex plane and therefore all the



quantities are real. This has the consequence of increased complexity in the formation of the function as an element or set of elements are required to be chosen for a particular solution. Note that the cyclic permutations of the  $e_{jk}$  elements form a closed group under multiplication which is isomorphic to quaternions[35, 55]. Eq. (53) is a source free solution to Maxwell's equations if it's four derivative is zero. This is demonstrated in Eq. (54) for the case where  $\omega = k$ .

$$d\phi = (k - \omega) (e_j + e_{0jk}) \exp[e_{jk}(kx_i - \omega t)] \quad (54)$$

These wave functions always require the appropriate selection of a replacement for the complex number  $i$ . Although this system is more complicated than the more conventional approach, it does have the advantage of being real everywhere. The cost of this advantage is a more complicated set of equations.

### 4.6.3 On the Polarisation of Light

In this section the polarisation of light and the different conventions which are utilised by different groups of people when describing the circular polarisation states (Right or Left) of a photon are introduced and discussed. In order to highlight this, a wave will be presented (See Figure 4.1) and interpreted it in terms of the different conventions; the reasons for the adoption of each particular convention will be noted and why each is of importance to the groups which utilise it.

The two most notable groups of people who utilise differing conventions for labelling the polarisation state of a photon are: those who work with light, primarily in the optics regime; and those who work with photons of particularly high energies i.e. high energy physicists (H.E.P). These two groups will consistently use different descriptive terms to denote the left or right handedness of any particular wave; except in the linearly polarised case.

The handedness of light for a (H.E.P) is defined by the way in which the incident light rotates the absorber. When a circularly polarised photon is absorbed, the absorbing body will gain angular momentum. The direction of this momentum (right = clockwise, left = anticlockwise) with respect to the momentum vector is then the handedness attributed to the (now non existent!) photon.

Conversely, those who work in optics place more emphasis and concern on

the field patterns made by the waves in space. Therefore they use a naming convention which describes the spatial twist of the wave, rather than any interaction properties. Both parties will agree on the mathematical description of the wave, the direction of the momentum vector and the projection of the angular momentum vector onto the momentum vector (positive or negative). They simply disagree on the labelling used for a particular wave. Table 4.1, shows how the various properties relate to the particular labels used by each group.

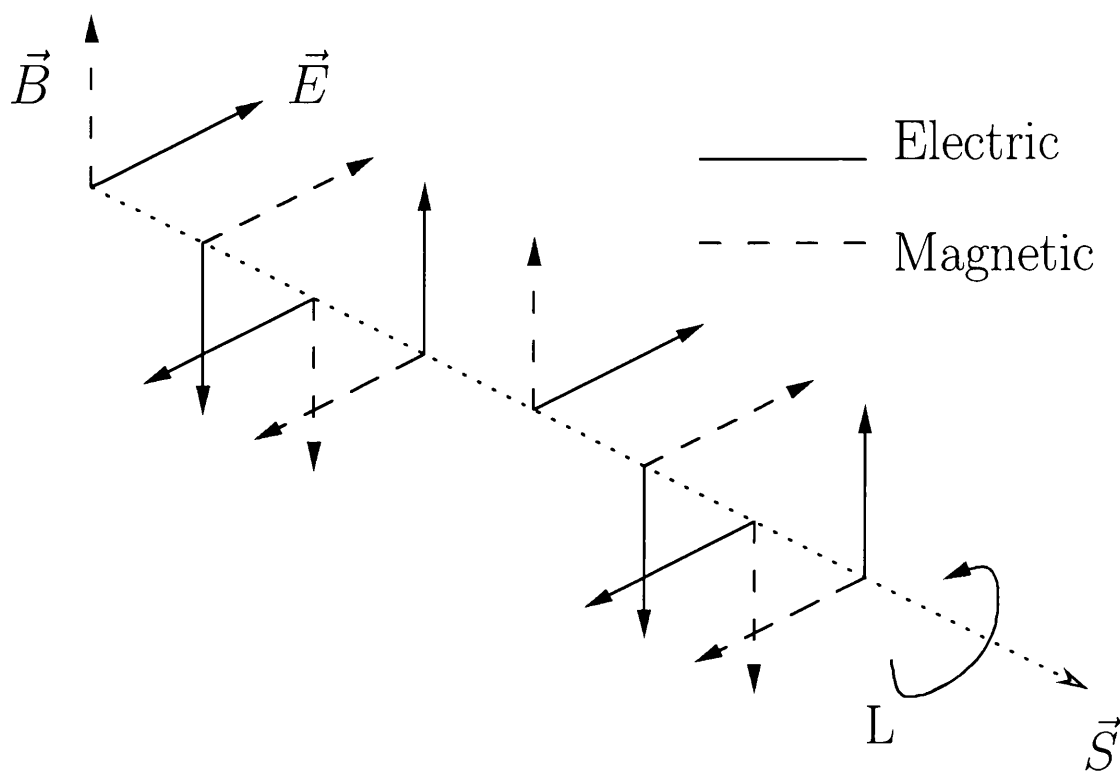
$\vec{S}$	$\vec{L}$	Prj	H.E.P	Optics
+	+	+	R	L
-	+	-	L	R
+	-	-	L	R
-	-	+	R	L

**Table 4.1.** This table shows the polarisation conventions for both high energy physicists (H.E.P) and optics. The “Prj” column refers to the projection of the angular momentum onto the momentum vector.

In this chapter and thesis, the optics convention is adopted when referring to the polarisation state of a wave function. This is partly because interactions will not be examined, but also because materials which utilise the optics convention will be referred to rather than the H.E.P convention.

#### 4.6.4 Polarisation States of Standard Wave Functions

This next section is concerned with addressing the polarisation states of the standard wave functions presented above[38]. As mentioned briefly in Section 4.6.2), “Standard Wave Functions”, the position of the field tensor arguments in the standard set of wave functions with respect to the propagation operator is crucial within this non-commutative algebra. The consequences of reordering the terms within the equation will be investigated here. By selective placement of the field tensor arguments to the left or right of the propagator, one can effectively change the overall sign of one of the field types (electric or magnetic) without altering its counterpart. The intention of this work is to provide the reader with a flavour for the kinds of manipulations which can be performed to



**Figure 4.1.** Diagram showing light with positive helicity and positive projection of the angular momentum vector ( $\vec{L}$ ) on the momentum vector ( $\vec{S}$ ).  $\vec{E}$  and  $\vec{B}$  represent the electric and magnetic fields respectively.

achieve different polarisations and phases; but also to demonstrate that different polarisation states can be achieved through linear superposition of states within this framework. Discussions and investigations into shaped waves (spherical, etc) will not be presented here.

In this section spatial position variables are denoted as  $x_i$  over cyclic permutations. A left\* circularly polarised wave, propagating in the  $\hat{k}$  direction is represented: -

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\*For a discussion on the polarisation naming conventions, see Section 4.6.3, "On the Polarisation of Light"

$$\phi_{L,Circ,+} = F_0 \exp [e_{ij}(\omega t - kx_k)] e_{0i} (1 + e_{0k}) \quad (55)$$

$$= F_0 (e_{0i} - e_{ik}) \cos (\omega t - kx_k) + F_0 (e_{0j} - e_{jk}) \sin (\omega t - kx_k) \quad (56)$$

Here  $k$  represents the usual wave number. A wave with the opposite propagation direction ( $-\hat{k}$ ), but with the same polarisation state is written as: -

$$\phi_{L,Circ,-} = F_0 \exp [-e_{ij}(\omega t + kx_k)] e_{0i} (1 - e_{0k}) \quad (57)$$

Similarly, wave functions with the opposite helicity in both propagation directions are: -

$$\phi_{R,Circ,+} = F_0 \exp [-e_{ij}(\omega t - kx_k)] e_{0i} (1 + e_{0k}) \quad (58)$$

$$\phi_{R,Circ,-} = F_0 \exp [e_{ij}(\omega t + kx_k)] e_{0i} (1 - e_{0k}) \quad (59)$$

A linearly polarised photon can be produced by the normalised sum of one left and one right circularly polarised wave, each with the same direction of propagation.

$$\phi_{Lin,+} = \phi_{R,Circ,+} + \phi_{L,Circ,+} \quad (60)$$

$$= F_0 (\exp [-e_{ij}(\omega t - kx_k + \theta)] + \exp [e_{ij}(\omega t - kx_k - \theta)]) e_{0i} (1 + e_{0k}) \quad (61)$$

$$= F_0 \exp [e_{ij}\theta] \cos (\omega t - kx_k) e_{0i} (1 + e_{0k}) \quad (62)$$

$$\phi_{Lin,-} = \phi_{R,Circ,-} + \phi_{L,Circ,-} \quad (63)$$

$$= F_0 (\exp [e_{ij}(\omega t + kx_k)] + \exp [-e_{ij}(\omega t + kx_k)]) e_{0i} (1 - e_{0k}) \quad (64)$$

$$= F_0 \exp [e_{ij}\theta] \cos (\omega t + kx_k) e_{0i} (1 - e_{0k}) \quad (65)$$

The consequences of replacing the chiral operator to the left of the propagator in the expression for the left circularly polarised wave, Eq. (55) above will now be examined. The wave function then becomes: -

$$\phi = e_{0i}(1 + e_{0k}) \exp [ e_{ij}(\omega t - kx_k) ] \quad (66)$$

By expanding Eq. (66), a description of the wave in terms of sines and cosines is once again obtained: -

$$\begin{aligned} \phi = & e_{0i} \cos(\omega t - kx_k) - e_{0j} \sin(\omega t - kx_k) \\ & - e_{ik} \cos(\omega t - kx_k) + e_{jk} \sin(\omega t - kx_k) \end{aligned} \quad (67)$$

Eq. (67) is now a right circularly polarised wave function but has the same momentum vector as Eq. (55), and therefore the same direction of propagation. The differences in these two (very similar) wave functions have arisen due to the sign change brought about as a result of the reversal in the ordering of multiplication. Due to the fact that switching the positions of the chiral and propagation operator has the effect of changing the polarisation state of the wave, a convention in which the chiral operator is always to the right of the propagator is adopted.

#### 4.6.5 New Wave Functions

In this section a new mathematical method for representing wave functions will be presented. These new wave functions reduce to the standard wave functions when certain conditions are fulfilled. The conditions, rather than the final wave functions representation, are the primary interest here. The example presented in this section is identical in representation to the circularly polarised wave function given in the previous section, however, the mathematical construction of the equation differs. Eq. (68) below is, for particular choices of  $\phi_0$ , a propagating transverse wave solution to the field equations.

$$\phi = \exp [(e_{0123}kx_1 - e_{23}\omega t)] \phi_0 \quad (68)$$

In Eq. (68),  $\phi_0$  is chosen such that  $|E| = |B|$  and  $\vec{E} \perp \vec{B} \perp \vec{S}$  where  $\vec{S}$  is the direction of propagation. Expanding the following equation: -

$$\phi = \exp [(e_{0123}kx_1 - e_{23}\omega t)] e_{02} (E_0 + B_0 e_{01}) \quad (69)$$

Gives: -

$$\begin{aligned}
&= e_{02}(E_0 \cos(kx_1) \cos(\omega t) - B_0 \sin(kx_1) \sin(\omega t)) \\
&\quad - e_{03}(B_0 \sin(kx_1) \cos(\omega t) + E_0 \cos(kx_1) \sin(\omega t)) \\
&\quad + e_{31}(E_0 \sin(kx_1) \cos(\omega t) + B_0 \cos(kx_1) \sin(\omega t)) \\
&\quad + e_{12}(B_0 \cos(kx_1) \cos(\omega t) - E_0 \sin(kx_1) \sin(\omega t))
\end{aligned} \tag{70}$$

Then by application of the following compound angle trigonometric identities: -

$$\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) = \cos(\alpha + \beta) \tag{71}$$

$$\cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) \tag{72}$$

$$\sin(\alpha) \cos(\beta) + \sin(\beta) \cos(\alpha) = \sin(\alpha + \beta) \tag{73}$$

$$\sin(\alpha) \cos(\beta) - \sin(\beta) \cos(\alpha) = \sin(\alpha - \beta) \tag{74}$$

Eq. (69) can be put into the form: -

$$\begin{aligned}
\phi &= (e_{02} + e_{12}) \cos(kx_i - \omega t) \\
&\quad - (e_{03} - e_{31}) \sin(kx_i - \omega t)
\end{aligned} \tag{75}$$

Which is a travelling wave solution to the source free Maxwell's equations where  $|E| = |B|$  and  $\vec{E} \perp \vec{B} \perp \vec{S}$ .

Examining the propagation operator,  $\phi_{prop}$ , for the new wave function: -

$$\phi_{prop} = \exp[(e_{0123} kx_i - e_{23} \omega t)] \tag{76}$$

Clearly shows that it does not describe rotations in a plane whereas those of the standard wave functions in Section 4.6.2 do.

$$\begin{aligned}
\phi_{prop} &= \cos(kx_i) \cos(\omega t) - e_{23} \cos(kx_i) \sin(\omega t) \\
&\quad + e_{0123} \sin(kx_i) \cos(\omega t) + e_{0i} \sin(kx_i) \sin(\omega t)
\end{aligned} \tag{77}$$

As has been shown above, the application of field elements to Eq. (76) & (77) and expansion reveals the terms which contribute to the travelling wave given in the previous section.

$$\begin{aligned} \phi_{prop} E_0 e_{02} = & E_0 e_{0j} \cos(kx_i) \cos(\omega t) - E_0 e_{03} \cos(kx_i) \sin(\omega t) \\ & + E_0 e_{31} \sin(kx_i) \cos(\omega t) - E_0 e_{12} \sin(kx_i) \sin(\omega t) \end{aligned} \quad (78)$$

$$\begin{aligned} \phi_{prop} B_0 e_{12} = & B_0 e_{12} \cos(kx_i) \cos(\omega t) + B_0 e_{31} \cos(kx_i) \sin(\omega t) \\ & - B_0 e_{03} \sin(kx_i) \cos(\omega t) - B_0 e_{02} \sin(kx_i) \sin(\omega t) \end{aligned} \quad (79)$$

The sum of Eq. (78) & (79) gives a travelling wave. For the case where  $\vec{E} \perp \vec{B}$ , four compound angle trigonometric identities are applied to the linearly dependent portions of the sum. For each identity, half the terms arise from Eq. (78) and half from Eq. (79). Hence only the sum of the two functions is a travelling wave. For the case where  $\vec{E} \parallel \vec{B}$  there is no identity which will put the dependent portions of the equations into travelling form. Hence only perpendicular portions of the fields propagate.

Notice that terms with the form of the magnetic field appear with magnitudes of the electric field  $E_0$  and vice versa. The magnitudes of the fields must be equivalent for successful application of the compound angle identities. Thus the condition  $E_0 = B_0$  is imposed on these new wave functions. As a result these functions are not capable of representing a solo field (electric or magnetic) without a counterpart.

The result of applying parallel fields to the propagation direction of  $\phi_{prop}$  are presented in Eq. (80) & (81) below.

$$\begin{aligned} \phi_{prop} e_{01} = & e_{01} \cos(kx_i) \cos(\omega t) - e_{0123} \cos(kx_i) \sin(\omega t) \\ & + e_{23} \sin(kx_i) \cos(\omega t) + \sin(kx_i) \sin(\omega t) \end{aligned} \quad (80)$$

$$\begin{aligned} \phi_{prop} e_{23} = & e_{23} \cos(kx_i) \cos(\omega t) + \cos(kx_i) \sin(\omega t) \\ & - e_{01} \sin(kx_i) \cos(\omega t) + e_{0123} \sin(kx_i) \sin(\omega t) \end{aligned} \quad (81)$$

These equations contain scalar and quadrivector components as well as fields parallel to the propagation direction. As is the case for parallel fields, there is no compound angle identity which will put Eq. (80) & (81) into travelling form. As the elements which result from this product are linearly independent of all other applicable fields, there is no potential for mixing parallel and non-parallel fields in an effort to find a combination which propagates. In these new wave functions portions of the fields parallel to the propagation direction do not propagate. Only for the case where the prefactors correspond to the argument of the exponential does the function describe a travelling wave. This means that there are strong extra constraints on the the choice of fields in the mathematical function. These constraints correspond exactly to the case where the fields propagate in nature.

There are certain issues when attempting a vector potential description in this format. Let  $\Psi$  be a wave description of a vector potential such that  $d\Psi$  is a wave description of the fields: -

$$\Psi = A \exp [e_{\ominus} (kx - \omega t)] \quad (82)$$

Where: -

$$A = e_{\mu} A_{\mu} \text{ (Summation)} \quad (83)$$

For the standard set of wave functions presented in Section 4.6.2, Eq. (82) is the general format of a vector potential wave description  $\Psi$ . In Eq. (82), suitable choices for  $e_{\mu} A_{\mu}$  and  $e_{\ominus}$  can be found such that  $d(d\Psi)$  is a non-trivial solution to the field equations. A vector potential description, equivalent to Eq. (82), in the format of the new wave functions presented in Section 4.6.5 is now sought.

$$\Psi_{new} = \left( e_0 A_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} \right) \exp [(e_{0123} kx_i - e_{23} \omega t)] \quad (84)$$

The propagator component of Eq. (84), as seen in Eq. (76), consists of scalar, bivector and quadrivector components depending on the phase. As a result, the



Einstein sum of  $e_\mu A_\mu$  in Eq. (84) will give rise to additional trivector components over the domain of the function. Attempts to cancel these components by introducing trivector arguments fail for phases where the function is purely scalar. Therefore this mathematical method, whilst able to reduce to a function which describes the fields, cannot be used to describe the potentials. The standard potential description must be used instead.

## 4.7 Discussion

In this chapter both the source free and inhomogeneous Maxwell's equations have been introduced and presented. Notice that as a consequence of using the four dimensional Clifford algebra, these equations appear at once, with all the correct signs, without requiring the introduction of a separate dual field strength tensor (See Section 4.5, "Dual Field Strength Tensor"). The requirement for introduction of the dual field strength tensor in the conventional formalisms[30] has been shown; illustrating that only half the terms are found otherwise. The fields equations have, however, also been presented in this way by Gull *et al* [17, 31].

The form of the fields have been derived through the choice of the four-potential as a point of origin. The bivector form of the electric field, whilst perhaps both unfamiliar and unintuitive, has followed naturally from the choices made in defining the spacetime algebra. A correspondence between the derived space-like bivector form for the magnetic field can more readily be made with the usual axial vector. The form which these fields take here is crucial to the properties and behaviour of the fields under Lorentz transformations (See Chapter 5, "Lorentz Transformations"). However, as was shown in Chapter 2 the spacetime-like bivector  $e_{0i}$  projects to a three-vector in the more conventional Euclidean algebras, which can result in a sign change in the exterior product.

In addition, some of the familiar types of wave functions used in other formalisms[30] have been introduced and their properties expanded into functions in the algebra presented in Chapter 2 (Section 2.6, "The Spacetime Algebra"). The complex form of the wave functions has been taken over by the introduction of a new form of the propagator. The form of the elements employed to replace the unit imaginary in this propagator have been shown to be isomorphic to quaternions (See

Section 2.9, “Properties of the Spacetime Algebra”). The resultant wave functions are purely real, describe both the electric and magnetic fields in a single equation, and integrate all the required properties one would expect. The drawback with the equations presented here is that they are more complicated; the wave functions are more intricate and require more care and attention in their construction than those which utilise complex numbers[30]. Note that Kramers[69] has also presented a composite wave function for both fields, these wave functions do, however, utilise complex numbers and are therefore not a purely real representation.

The new wave functions presented in this chapter have the unusual property that they are only travelling wave solutions under certain conditions. These conditions are that magnitudes of the fields must be equivalent and that  $\vec{E} \perp \vec{B} \perp \vec{S}$  where  $\vec{S}$  represents the propagation direction. These new wave functions are incapable of propagating a solo electric or magnetic fields without a counterpart and with a Poynting vector of zero[83]. The format of these new functions essentially place algebraic constraints on solutions which excludes the light speed propagation of certain non-physical fields. The disadvantage with these new wave functions is that as the magnitudes of the electric and magnetic field must be the same at all points in the function, the formation of polarisation states is restricted somewhat.

Attempts to form a wave descriptions of the vector potential using a wave function of this format has demonstrated that vector argument give rise to trivector components over the domain of the function. Attempts to cancel these components have proved to be in vain. Therefore the format of the new functions prohibit origins in a purely vector potential description within the formalism presented here. Although the use of a secondary potential has been implemented successfully by other authors[84] as a means of introducing the angular momentum in the field, the approach is not pursued here.

## 4.8 Chapter Summary

In this chapter the behaviour and utility of the spacetime algebra in describing relativistic electromagnetism has been presented. This investigation began by introducing quantities such as the vector potential. This was used to introduce

and derive the form of the field tensors, both of which appear as bivectors. This was contrasted with other approaches and the consequences of any differences explained.

In particular Maxwell's equations were introduced using the algebra presented in Section 2.6, "The Spacetime Algebra". This derivation highlights the fact that all the terms appear at once, with the correct signs and with no need to introduce a separate dual field strength tensor. The familiar form of the dual field strength tensor  $F^{\alpha\beta}$  in the usual relativistic notation has been introduced. Showing that without the introduction of this tensor, only half the field terms appear in the derivation of Maxwell's equations.

The gauge degree of freedom has been investigated in this chapter and identified with the scalar portion of the derivative of the four potential  $A$  (See Section 4.2, "Potentials & Fields"). How simplifications can be made to the dynamical field equations by setting various portions which contribute to this scalar has been shown. These simplifications are well known to lead to solutions of the field dynamics[81, 82]. Two common gauge conditions have been introduced in the context of the algebra presented here, the difference between gauge conditions and the gauge itself have been illustrated.

This work on the gauge has then been utilised as a basis for the analysis of a speculative gauge condition. This new gauge condition  $dL = -J$  has been shown to reduce the dynamics of the fields to a homogeneous wave equation in terms of the four-potential. In addition, the wave equation supports the introduction of current densities and charge densities through a scalar field (See 4.4, "The Gauge").

Wave function solutions to the field equations have been introduced and methods and techniques for replacing the complex number  $\mathbf{i}$  with a completely real system have been demonstrated. Provided that a suitable replacement for  $\mathbf{i}$  is utilised, the electric and magnetic field solutions can be represented in a single purely real function.

The polarisation conventions utilised by various groups have also been introduced and discussed in this chapter. The naming criteria for two conventions, optics and high energy physics, have been presented and based on this discussion a choice to adopt the optics convention in this thesis has been made. Base polarisation states for wave functions, including left, right and linearly polarised

waves in both propagation directions have been presented and discussed. The significance in the ordering of the field argument (chiral operator) with respect to the propagator term has been shown to be crucial in the non-commutative algebra, in that it reverses the helicity (but not the momentum vector) of the wave. Therefore, for the sake of convention and readability, the chirality operator is always to be placed to the right of the propagator[38].

Finally a new set of wave functions, which have no analogy in the usual relativistic notation, have been introduced and examined. Although upon first inspection, these functions do not appear to support travelling wave solutions, they are in fact capable of propagating a limited set of real and physical fields. These functions have been subjected to an analysis in order to ascertain the extent of these limitations. These functions have been shown to be equivalent to the standard wave functions for cases where the propagation operator  $\phi_{prop}$  and the fields match such that  $\vec{E} \perp \vec{B} \perp \vec{S}$  where  $\vec{S}$  is the propagation direction, and the electric and magnetic fields are of the same magnitude.

# Chapter 5

## Lorentz Transformations

### 5.1 Introduction

In this chapter the Lorentz transformation and Lorentz covariance within the algebra presented in Chapter 2 will be examined. The motivation for exploring these transformations is to ensure that the field equations presented in Chapter 4 are relativistically covariant. The Lorentz transformation will be introduced in the following section and will be initially put forth as an extension of the classical Galilean transform[19, 20]. The aim of this chapter is to provide a more detailed analysis of Lorentz transformations and covariance within this algebra than has been presented in the past. To this end, the work of Hestenes[29, 34], Lounesto[27] & Gull *et al*[17] will be examined. This will be used to present Lorentz transformations in the Clifford algebra  $\mathcal{Cl}_{1,3}$ .

Attention will be drawn to the way in which physical quantities and elements of the algebra (Section 2.6, “The Spacetime Algebra”) behave under Lorentz transformation. The covariance of the field wave function solutions presented in Chapter 4 will be demonstrated. Particular attention is paid to the energies of these wave functions under transformation and comparisons will be drawn with experimental evidence. Discrepancies will be discussed and resolved where possible.

Relativistic measurements in the inertial rest frame will be presented in this chapter. The definitions and concepts which are “proper time” or, as it sometimes referred to, “world time” will be given[78, 19, 85, 86, 87]. Proper quantities will

be put forth in a similar fashion[29]. The form for the differential operator under Lorentz transformation will be examined and discussed.

Later in this this chapter Lorentz invariance will be introduced and discussed[20]. Quantities which remain the unchanged under Lorentz transformations are Lorentz invariants. Certain physical quantities, which are known to remain unchanged under a Lorentz transformation, will be obtained and compared to the work of Morse *et al.* Similarly, the lack of a preferred reference frame for the description of physical laws will be shown. General invariant properties of multivector quantities will be derived and proven.

Finally, the covariance of energy-momentum quantities using this algebra will be examined and discussed. The form of the energy momentum tensor will be presented. Discussions by Boyer[21, 88], Rohrlich[22, 23] and others[89, 90] will then be examined in the context of the algebra presented here.

## 5.2 The Lorentz Transformation

### 5.2.1 Introduction

An inertial frame transformation in which time is absolute is an example of a Galilean transformation. Such transformations are non-relativistic[7] and are sometimes referred to as Newtonian transformations[91]. These transformations belong to non-relativistic “classical mechanics”.

In cases where the velocity is small compared to that of the speed of light, the Galilean transforms approximates closely to the results of special relativity[7, 78]. Rotations in this non-relativistic system only take place in the three spatial dimensions and time is treated as a parameter. No physically possible operation, within this framework, is capable of rotating the time axis into a space axis or vice versa[78]. The special theory of relativity[7] has demonstrated, however, that when the relative velocity of two observers is a significant fraction of the speed of light their time directions are measurably not parallel[78]. Feynman[92] conveys this as “the space measurements of one man is mixed in a little bit of time, as seen by another”. As discussed in Section 2.9.2, “Minkowski Spacetime”, Minkowski[39, 40] was the first to introduce the concept of a spacetime in which space and time are treated as being intimately related with each other, instead

of being separate entities.

The Lorentz transform is a mathematical method for representing all space-time rotations and consequently determining observables in different inertial frames. These transformations are proper spacetime transformations from one inertial frame to another. This section deals with relativistic transformations to different inertial frames (i.e. frames moving with uniform velocity); the study of accelerating frame transformations and gravitation is beyond the area of study of this thesis. A full Lorentz transformation can be separated in to inertial frame transformations, which will be referred to as Lorentz Boosts, and rotational transformations, which will be referred to as as Lorentz Rotations[54, 53]. Lorentz Rotations are introduced in Section 5.4, “Proper Rotations”.

The work presented in this Chapter is, with the exception of the introduction to the Lorentz transform, and the discussion of the Boyer-Rohrlich controversy is an original re-examination and rigorous investigation of Lorentz transformations in  $\mathcal{Cl}_{1,3}$ . This Chapter utilises the software developed in Chapter 3 to ascertain the conditions under which an element of the algebra transforms and these are related to some of the properties defined in Chapter 2. Using these relations, generalisations regarding the transformation properties of the algebra have been deduced.

## 5.2.2 Lorentz Boosts

In special relativistic systems[7, 39, 93], in which time is no longer taken to be absolute, length and time scales differ from the Galilean transformation by a factor  $\gamma$  given by: -

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \quad (1)$$

The history of this equation is rather curious; the Lorentz transform predates the special theory of relativity by a number of years and was developed as part of a classical theory of electrons[94, 95]; where  $v$  supposedly represents a velocity relative to the “ether”[19, 91].

Consider the case of two bodies A & B, A is moving with a uniform relative velocity to B along the x-axis. A Lorentz boost of the unprimed frame of A

$(t, x, y, z)$  corresponds to a rotation  $\theta$  in the  $(ct, x)$  plane[78, 34, 38]. This rotation is represented by: -

$$x' = x \cosh \theta + ct \sinh \theta \quad (2)$$

$$y' = y; \quad z' = z \quad (3)$$

$$ct' = x \sinh \theta + ct \cosh \theta \quad (4)$$

The primed coordinate set  $(t', x', y', z')$  corresponds the frame of body B. The relative velocity of the two bodies can expressed as a function of the angle  $\theta$ : -

$$\beta = \frac{v}{c} = \tanh \theta \quad (5)$$

where: -

$$\sinh \theta = \beta \gamma; \quad \cosh \theta = \gamma \quad (6)$$

In addition, if a third body (C) is introduced with a relative velocity  $v = c \tanh \delta$  with respect to B. Then the relative velocity of body C with respect to body A is  $[c \tanh (\theta + \delta) = \frac{(u+v)}{[1+(uv/c^2)]}]$ .

This technique for rotations between a spatial and the time axis will now be applied to the algebra presented in Chapter 2. Hestenes[34] has shown that the three-spacetime bivectors,  $e_{0i}$ , may be used as generators of rotations in a time-space plane. The form of the general Lorentz boost in this algebra is[34, 38]:

$$\Psi' = \kappa \Psi \kappa^{-1} \quad (7)$$

For a general Lorentz boost  $\kappa$  must be such that when squared Eq. (7) has the form: -

$$\kappa^2 = \cosh(\theta) + e_{0i} \sinh(\theta) \quad (8)$$

When multiplied by a four-vector, Eq. (8) yields a transformed coordinate system. When a unit vector is multiplied by Eq. (8) the unit vectors transform as: -



$$\hat{x}'_i = e'_i = [e_i \cosh(\theta) + e_0 \sinh(\theta)] \quad (9)$$

$$\hat{t}' = e'_0 = [e_0 \cosh(\theta) + e_i \sinh(\theta)] \quad (10)$$

which is the same as the first example. Once again  $\theta = \tanh^{-1}\beta$  and  $\beta = v/c$ . The general form of  $\kappa$  is[34]: -

$$\kappa = \exp \left[ e_{0i} \frac{\theta}{2} \right] \quad (11)$$

As each  $e_{0i}$  element squares to unity Eq. (11) will expand in terms of hyperbolic sinusoids. In Eq. (7), all elements of  $\Psi$  which commute with the non-scalar portion of  $\kappa$  will remain untransformed (as  $\kappa\kappa^{-1} = 1$ ) and all elements which anti-commute will be transformed by the factor  $\kappa^2 = \exp[e_{0i}\theta]$ . For the case of unit vectors, expressions similar in form to Eq. (9) & (10) will result. This transformation procedure selectively determines which portions of quantities are transformed based on the commutation relations of the elements with respect to the boost element ( $e_{0i}$ ).

Any general multivector can be expanded and expressed in terms of a sum of those portions which commute and those which anti-commute with a particular basis element. i.e.  $\Psi = \Psi_C + \Psi_A$ . For convenience the commuting part is denoted as  $\Psi_C$  and the anti-commuting part as  $\Psi_A$ . In this instance where any component of  $\Psi$  commutes with the non-scalar portion of  $\kappa$ , the component forms part of  $\Psi_C$ . Conversely, if a component anti-commutes then component will form part of  $\Psi_A$ . The commuting quantities are invariant and the anti-commuting quantities vary as  $\kappa^2 = \exp[e_{0i}\theta]$  as shown below: -

$$\Psi' = \kappa^2 \Psi_A + \Psi_C \quad (12)$$

As  $\Psi_C$  is the portion of  $\Psi$  which commutes with the non-scalar portion of  $\kappa$ , the factor  $\kappa\kappa^{-1}$  vanishes from the term. Similarly for  $\Psi_A$ , the anti-commute of  $\kappa$  changes the sign of non-scalar portion of  $\kappa$  but does not change the sign of scalar portion. The result is a change of sign in the argument of the exponential, effectively changing  $\kappa^{-1}$  to  $\kappa$  as the partial anti-commutation takes place. The resulting expression is given in Eq. 12 which shows that only the anti-commuting portions of  $\Psi$  will be transformed. The precise behaviour of each element can be

found by examining the outcome when each basis element has this transformation process applied. The software developed in Chapter 3 has been used to perform the mechanical calculations rapidly and accurately.

Boost	Element	Even/Odd	$\perp, \parallel$	Boosted?	Prediction	Boosted Function
$e_{0i}$	$\mathbb{1}$	E	U	NO	-	$(\mathbb{1})$
$e_{0i}$	$e_0$	O	U	YES	-	$(\gamma e_0 - \beta \gamma e_i)$
$e_{0i}$	$e_i$	O	$\parallel$	YES	YES	$(\gamma e_i - \beta \gamma e_0)$
$e_{0i}$	$e_j$	O	$\perp$	NO	NO	$(e_j)$
$e_{0i}$	$e_k$	O	$\perp$	NO	NO	$(e_k)$
$e_{0i}$	$e_{0i}$	E	$\parallel$	NO	NO	$(e_{0i})$
$e_{0i}$	$e_{0j}$	E	$\perp$	YES	YES	$(\gamma e_{0j} - \beta \gamma e_{ij})$
$e_{0i}$	$e_{0k}$	E	$\perp$	YES	YES	$(\gamma e_{0k} + \beta \gamma e_{ki})$
$e_{0i}$	$e_{jk}$	E	$\parallel$	NO	NO	$(e_{jk})$
$e_{0i}$	$e_{ki}$	E	$\perp$	YES	YES	$(\gamma e_{ki} + \beta \gamma e_{0k})$
$e_{0i}$	$e_{ij}$	E	$\perp$	YES	YES	$(\gamma e_{ij} - \beta \gamma e_{0j})$
$e_{0i}$	$e_{0jk}$	O	$\parallel$	YES	YES	$(\gamma e_{0jk} - \beta \gamma e_{123})$
$e_{0i}$	$e_{0ki}$	O	$\perp$	NO	NO	$(e_{0ki})$
$e_{0i}$	$e_{0ij}$	O	$\perp$	NO	NO	$(e_{0ij})$
$e_{0i}$	$e_{123}$	O	U	YES	-	$(\gamma e_{123} - \beta \gamma e_{0jk})$
$e_{0i}$	$e_{0123}$	E	U	NO	-	$(e_{0123})$

**Table 5.1.** Table showing the behaviour of the individual basis elements under a Lorentz transformation (boost). In the  $\perp, \parallel$  column, U is used when the direction of the element under transformation is undefined with respect to the boost element. See Section 2.8, “Parallel Objects”.

Table 5.1 shows the properties of the basis elements under Lorentz transformations. There is a symmetry between whether a quantity is invariant, the grade of the element (odd or even) and the perpendicularity of that element with respect to the boost itself. The transformation of an odd element occurs only in the case where the element is parallel to the boost. The transformation of an even element occurs only if the element is perpendicular. There are four special cases where the direction of the element with respect to a  $e_{0i}$  element is undefined. These are marked with a U in the table and they are the scalar, spacetime-like vector ( $e_0$ ), space-like trivector ( $e_{123}$ ) and the pseudoscalar ( $e_{0123}$ ). In the case of these four, the even ones ( $\mathbb{1}$  &  $e_{0123}$ ) are invariant under all Lorentz boosts and the odd ones ( $e_0$  &  $e_{123}$ ) are always transformed under a boost. For a more

details on the definitions of parallel objects within the algebra see Section 2.8, “Parallel Objects”.

Table.5.1 shows that the Lorentz boost will produce a scalar component only from the scalar itself and also that the scalar is invariant under Lorentz transforms. Therefore *the scalar component of any multi-vector  $\Psi$  is the same in all inertial frames*. This is also true of the pseudoscalar  $e_{0123}$ . The scalar is a true Lorentz scalar in the Einstein sense. This has consequences for what may, and may not be considered a proper scalar quantity within this algebra. This shall be discussed in more detail later.

### 5.2.3 Wave Functions

The behaviour of wave functions under Lorentz transformation will now be examined. Consider a circularly polarised wave function with the momentum vector in the  $+\hat{x}_k$  direction\*: -

$$\phi_{circ} = \exp [e_{ij}(kx_k - \omega t)] e_{0i} (1 + e_{0k}) \quad (13)$$

The application of Eq. (7) & (11) is used to transform Eq. (13). The transformations correspond to fractional changes in relative velocity of the wave compared to the speed of light, i.e  $\beta c$  where  $\beta = 0 \rightarrow 1$ . In Eq. (11),  $\theta$  becomes  $\tanh^{-1}(\beta)$  (natural units). For example a boost in the longitudinal ( $\hat{x}_k$ ) direction is to be applied therefore  $e_{0k}$  is the appropriate choice for boost element in Eq. (11). The transformed wave is: -

$$\phi'_{circ} = \exp \left[ e_{0k} \frac{\tanh^{-1}(\beta)}{2} \right] \phi_{circ} \exp \left[ - e_{0k} \frac{\tanh^{-1}(\beta)}{2} \right] \quad (14)$$

As  $\phi_{circ}$  is a multivector object, which may contain terms which both commute and anti-commute with  $\kappa$ , the calculations are simplified by the separation these of these constituent components ( $\phi_C$  and  $\phi_A$ ). Table 5.1 is utilised for this separation process. In the case of Eq. (14), all the terms form part of  $\phi_C$  and  $\phi_A$  is zero. The transformed expression is therefore: -

---

\*Recall from Section 2.6.8 that “exp” is used to denote exponentials rather than “e”, as the latter may invoke confusion with basis elements.

$$\begin{aligned}\phi'_{circ} &= \kappa^2 \phi_{circ} & (15) \\ &= (\gamma - e_{0k}\beta\gamma) \left( (e_{0i} + e_{ki}) \cos(kx_k - \omega t) + (e_{0j} - e_{jk}) \sin(kx_k - \omega t) \right)\end{aligned}$$

As an example, consider the case where  $\beta = 0.5$ , Eq. (15) yields  $\gamma = 1.154$  and  $\beta\gamma = 0.577$  indicating an overall reduction in the field energy ( $U = 1/2(E^2 + B^2)$ ). The energy density, for  $\beta = 0.5$ , of this wave has reduced to a  $1/3$  of that of the untransformed wave. A modified Doppler shift equation can be used to compare this with experiment: -

$$U' = \hbar\omega' = \frac{\hbar\omega_0(1 - v/c)}{\sqrt{1 - v^2/c^2}} = \hbar\omega_0\gamma(1 - \beta) \quad (17)$$

This Doppler shift equation has been modified to take into account the time dilation of the source[92]\*. The energy of the transformed wave for Eq. (15) is: -

$$U' = \frac{\gamma}{2} \hbar\omega_0 \quad (18)$$

which is  $U = 0.577 \hbar\omega_0$  for  $\beta = v/c = 0.5$  (See Eq. (1) ) and thus disagrees with the values obtained from the Lorentz transformation in Eq. (15).

The energy is, however, covariant when the wave functions are taken as being representative of densities. Due to the volume of the wave transforming along with the fields, the energy must be computed in terms of a volume integral over the new volume in order to obtain the correct value for the boosted energy.

Assume that the geometry of the wave is cylindrical, with a radius of  $\lambda/2$  perpendicular to the momentum vector. The longitudinal axis corresponds to one wavelength  $\lambda$ . The energy of the wave function is represented, in the untransformed case, by the integral: -

$$U = \frac{2\pi}{2} \int \int (\vec{E}^2 + \vec{B}^2) \partial\lambda \partial r^2 \quad (19)$$

$$= \frac{\pi}{2} (\vec{E}^2 + \vec{B}^2) \lambda_0 r^2 \quad (20)$$

---

\*Note that this calculation makes the assumption that the mean velocity of the source was zero in the original frame. If this is not the case then a different velocity for  $\gamma$  is required.

The direction of the boost, and the length contraction/expansion, is parallel to the momentum vector of the wave. Here  $\lambda_0$  is substituted for the new elongated wavelength. This wavelength is  $\lambda' = \frac{1}{\gamma(1-\beta)}\lambda_0$ . A relativistic Doppler shift equation is used once again. Performing the substitution into Eq. (20).

$$U' = \frac{\pi}{2} (\gamma - \beta\gamma)^2 (\vec{E}^2 + \vec{B}^2) \frac{1}{\gamma(1-\beta)} \lambda_0 r^2 \quad (21)$$

$$= \frac{(\gamma - \beta\gamma)^2}{(\gamma - \beta\gamma)} U \quad (22)$$

$$= (\gamma - \beta\gamma) U \quad (23)$$

$$(24)$$

In Eq. (21), for  $\beta = 0.5$  and  $\gamma = 1.732$ ,  $U' = 0.577 U$  which agrees with the value calculated from the relativistic Doppler shift in Eq. (18).

The properties of this wave function (Eq. (13)) will be examined under a boost in the perpendicular  $\hat{x}_i$  direction and compared to the (parallel) transformation above. The separation of the terms which commute and anti-commute with the transformation function is required. This transformation function is now  $\kappa = \exp \left[ e_{0i} \frac{\tanh^{-1}(v_{rc})}{2} \right]$ : -

$$\phi_C = e_{0i} \cos(kx_k - \omega\hat{t}) - e_{jk} \sin(kx_k - \omega\hat{t}) \quad (25)$$

$$\phi_A = e_{ki} \cos(kx_k - \omega\hat{t}) + e_{0j} \sin(kx_k - \omega\hat{t}) \quad (26)$$

From Eq. (25) & (26), only the fields in the  $\hat{x}_j$  will be transformed. The  $\hat{x}_i$  fields are invariant under this particular transformation. The transformed function now becomes: -

$$\begin{aligned} \phi'_{circ} &= e_{0i} \cos(kx_k - \omega\hat{t}) - e_{jk} \sin(kx_k - \omega\hat{t}) \\ &+ \gamma (e_{ki} \cos(kx_k - \omega\hat{t}) + e_{0j} \sin(kx_k - \omega\hat{t})) \\ &+ \beta\gamma (e_{0k} \cos(kx_k - \omega\hat{t}) - e_{ij} \sin(kx_k - \omega\hat{t})) \end{aligned} \quad (27)$$

This transform has generated fields parallel to the momentum vector of the

wave in Eq. (27). For  $\beta = 0.5$ , the overall energy density increase is exactly  $2\frac{2}{3}$  of the original value. Regardless of the sign of the boost, the energy density of the system will increase by a factor of  $2\frac{2}{3}$ .

These transformed functions are no longer solutions of the homogeneous Maxwell's equations ( $dF = 0$ ) when measured in the untransformed frame. The four derivative of Eq. (27) is: -

$$\begin{aligned}
d(\phi'_{circ}) = & \omega (e_{0jk} \cos(kx_k - \omega \hat{t}) - e_i \sin(kx_k - \omega \hat{t})) \\
& - k (e_{0ki} \sin(kx_k - \omega \hat{t}) + e_j \cos(kx_k - \omega \hat{t})) \\
& + \gamma \omega (e_{0ki} \sin(kx_k - \omega \hat{t}) + e_j \cos(kx_k - \omega \hat{t})) \\
& + \beta \gamma \omega (e_k \sin(kx_k - \omega \hat{t}) - e_{0ij} \cos(kx_k - \omega t)) \\
& + \gamma k (e_i \sin(kx_k - \omega \hat{t}) - e_{0jk} \cos(kx_k - \omega \hat{t})) \\
& + \beta \gamma k (e_0 \sin(kx_k - \omega \hat{t}) - e_{123} \cos(kx_k - \omega \hat{t}))
\end{aligned} \tag{28}$$

Eq. (28) cannot be zero in any non-trivial manner. If the differential operator is also transformed: -

$$d' = \kappa d \kappa^{-1} = \frac{\partial}{\partial t} (\gamma e_0 + \beta \gamma e_1) - \frac{\partial}{\partial x} (\gamma e_1 + \beta \gamma e_0) - \frac{\partial}{\partial y} e_2 - \frac{\partial}{\partial z} e_3 \tag{29}$$

the measurements are equivalent to those taken in the primed frame by the untransformed differential. This can be confirmed by examining the four differential of  $\phi'_{circ}$  using this transformed differential operator is: -

$$d'(\phi'_{circ}) = 0 \tag{30}$$

Expressing the Lorentz covariance of the measurement[20]. The transformed differential operator is a form of the covariant derivative and is the derivative as seen in the rest frame of an object. The covariant derivative is examined in more detail in the following section.

## 5.3 Proper Quantities & The Covariant Derivative

Proper time, proper lengths and proper quantities will be introduced in this section. An observable quantity is “proper” when it is equivalent to a measurement of that quantity in the rest frame of the object under examination[78, 19, 85, 86, 87]. The length of a body, for example, as measured in it’s rest frame is called it’s proper length. Any other measurement of the length is “non-proper” and will differ from the proper length by a factor of  $\gamma^{-1}$  (See Eq. (1)). Similarly “proper time” is time as measured at a fixed point in the rest frame of an object[86]. The measurements of a time interval between two events is non-proper unless the both events are recorded in terms of two clocks known to be synchronised with each other.

The covariant derivative will also be examined in this section. This derivative can also be thought of as the “proper derivative”. i.e. the derivative as measured in the rest frame of the object under examination. The covariant derivative, as presented here, is defined as a standard four derivative, which has been transformed into the rest frame of an object under examination. This in effect, altered the rulers and clocks in such a way that they correspond to what is observed in a different inertial frame.

Recall the quote from Feynman’s lectures in the introduction, “*the space measurements of one man is mixed in a little bit of time, as seen by another*”. Closer examination of a transformed differential operator: -

$$d' = \frac{\partial}{\partial t} (\gamma e_0 + \beta\gamma e_1) - \frac{\partial}{\partial x} (\gamma e_1 + \beta\gamma e_0) - \frac{\partial}{\partial y} e_2 - \frac{\partial}{\partial z} e_3 \quad (31)$$

Eq. (31) illustrates that the space and time measurements do mix in a covariant way. The second derivative does not transform to a covariant form due to it’s scalar nature. This is true of all even power derivatives (i.e  $d^2$ ,  $d^4$ ,  $d^6$  etc). The second derivative of Eq. (27): -

$$\square = (d')^2 = \kappa d \kappa^{-1} \kappa d \kappa^{-1} = \kappa d^2 \kappa^{-1} = \kappa \kappa^{-1} d^2 \quad (32)$$

As  $d^2$  is a scalar, which commutes with  $\kappa$ , there is no transformation. The application of this to Eq. (27): -

$$\square(\phi'_{circ}) = 0 \quad (33)$$

Eq. (33) illustrates that the second order Maxwell equations are invariant under Lorentz transformation in this algebra. The covariant derivative is only required in electromagnetism if one wishes to examine the proper currents in a transformed frame. Higher order derivatives can also be found in a similar way; for example the third order derivative: -

$$d^3 = (d')^3 = \kappa d \kappa^{-1} \kappa d \kappa^{-1} \kappa d \kappa^{-1} = \kappa^2 d^3 \quad (34)$$

The general rule for Nth order derivatives is that even ordered derivatives are invariant and that the odd ordered derivatives are covariant.

## 5.4 Proper Rotations

### 5.4.1 Introduction

To complete the discussion of general Lorentz transformations within this algebra Proper rotations, otherwise known as Lorentz rotations[34], will be presented in this section. These transformations will be examined utilising the same procedures applied to the case of the Lorentz boosts. The equations will be introduced and the properties of the basis elements under rotation will be examined. The investigation will then focus on the transformation properties of wave equations.

### 5.4.2 Lorentz Rotors

The form of a general Lorentz rotation in this algebra is[34]: -

$$\Psi' = R\Psi R^{-1} \quad (35)$$

For a general Rotation  $R$  must be such that when squared: -

$$R^2 = \cos(\theta) + e_{ij} \sin(\theta) \quad (36)$$

Eq. (36), when multiplied into a four vector yields a coordinate system rotated by an angle  $\theta$  in the  $i, j$  plane. Therefore  $R$  is: -



$$R = \exp \left[ \frac{e_{ij}\theta}{2} \right] \quad (37)$$

Each  $e_{ij}$  element square to minus unity, therefore this function will expand in terms of periodic sinusoids. In Eq. (35), all elements of  $\Phi$  which commute with  $R$  will not be rotated (as  $RR^{-1} = 1$ ) and all elements which anti-commute will be multiplied by  $R^2 = \exp [e_{ij}\theta]$ . (See Section 5.2

$$\Psi' = R^2\Psi_A + \Psi_C \quad (38)$$

where  $\Psi_C$  and  $\Psi_A$  represent the commuting and anti-commuting portions of the multivector  $\Psi$  (with respect to the non-scalar portion of  $R$ ) respectively. The precise behaviour of the basis elements under a proper rotation (i.e. whether or not they rotate) is found by examining the outcome of each basis element when this transformation is applied. An attempt is made to rotate each element through an angle  $\theta$ , in each of the three-space planes. The software developed in Chapter 3 is used, once again, to perform the calculations.

Table. 5.2 shows the behaviour of all the basis elements under Lorentz rotation. The covariance or otherwise of an element under rotation has no dependence on whether it is odd or even (See Section 2.6.5, “Even and Odd Basis Elements”). Therefore, from Table 5.2, *an element is only rotated if it is perpendicular to the axis of rotation*. There are four special cases where the direction of the element with respect to a  $e_{ij}$  element is undefined. These are marked with a U in the table and they are the scalar, spacetime-like vector ( $e_0$ ), space-like trivector ( $e_{123}$ ) and the pseudoscalar ( $e_{0123}$ ). All of these four elements are invariant under rotation. Aside from the elements themselves, no rotational operation produces a component of these four elements. For a more detailed description and discussion on the perpendicularity of basis elements with respect to one another see Section 2.8, “Parallel Objects”.

### 5.4.3 Wave Functions

The properties and behaviour of wave functions under Lorentz Rotation will now be examined. Consider a right circularly polarised wave travelling in the  $x_k$  direction: -

Rotor	Element	Even/Odd	$\perp, \parallel$	Rotated?	Prediction	Rotored Function
$e_{jk}$	$\mathbb{1}$	E	U	NO	-	$\mathbb{1}$
$e_{jk}$	$e_0$	O	U	NO	-	$e_0$
$e_{jk}$	$e_i$	O	$\parallel$	NO	NO	$e_i$
$e_{jk}$	$e_j$	O	$\perp$	YES	YES	$C\theta e_j + S\theta e_k$
$e_{jk}$	$e_k$	O	$\perp$	YES	YES	$C\theta e_k - S\theta e_j$
$e_{jk}$	$e_{0i}$	E	$\parallel$	NO	NO	$e_{0i}$
$e_{jk}$	$e_{0j}$	E	$\perp$	YES	YES	$C\theta e_{0j} + S\theta e_{0k}$
$e_{jk}$	$e_{0k}$	E	$\perp$	YES	YES	$C\theta e_{0k} - S\theta e_{0j}$
$e_{jk}$	$e_{jk}$	E	$\parallel$	NO	NO	$e_{jk}$
$e_{jk}$	$e_{ki}$	E	$\perp$	YES	YES	$C\theta e_{ki} + S\theta e_{ij}$
$e_{jk}$	$e_{ij}$	E	$\perp$	YES	YES	$C\theta e_{ij} - S\theta e_{ki}$
$e_{jk}$	$e_{0jk}$	O	$\parallel$	NO	NO	$e_{0jk}$
$e_{jk}$	$e_{0ki}$	O	$\perp$	YES	YES	$C\theta e_{0ki} + S\theta e_{0ij}$
$e_{jk}$	$e_{0ij}$	O	$\perp$	YES	YES	$C\theta e_{0ij} - S\theta e_{0ki}$
$e_{jk}$	$e_{123}$	O	U	NO	-	$e_{123}$
$e_{jk}$	$e_{0123}$	E	U	NO	-	$e_{0123}$

**Table 5.2.** Table showing the behaviour of the individual multi-vector quantities under a proper rotation. Here  $C\theta = \cos(\theta)$  and  $S\theta = \sin(\theta)$ . In the  $\perp, \parallel$  column, U is used when the direction of the element under transformation is undefined with respect to the rotor element.

$$\phi_{circ} = \exp [e_{ij}(kx_k - \omega t)] e_{0i} (1 + e_{0k}) \quad (39)$$

The properties of Eq. (39) under rotation in the  $\hat{x}_j - \hat{x}_k$  plane will now be examined. This transformation corresponds to an arbitrary rotation of  $\theta$  in a plane non-parallel with the momentum vector. The element  $e_{jk}$  is chosen as the replacement for  $e_{ij}$  in  $R$ . As  $\phi_{circ}$  is a multivector object which may contain terms which both commute and anti-commute with  $R$ , the calculations are simplified by the separation of these terms ( $\phi_C$  and  $\phi_A$ ). Expansion of the exponential terms is required as the associative rules do not hold for the arguments of exponential functions (See Section 2.6.8, "Sines, Cosines & Exponential Functions"). Therefore  $\phi_A$  and  $\phi_C$  are: -

$$\phi_A = e_{ki} \cos(kx_k - \omega\hat{t}) + e_{0j} \sin(kx_k - \omega\hat{t}) \quad (40)$$

$$\phi_C = e_{0i} \cos(kx_k - \omega\hat{t}) - e_{jk} \sin(kx_k - \omega\hat{t}) \quad (41)$$

where  $R = \exp [e_{jk} (\theta/2)]$  the transformed wave is: -

$$\phi'_{circ} = R^2 \phi_A + \phi_C \quad (42)$$

$$= \exp [e_{jk} \theta] \phi_A + \phi_C \quad (43)$$

$$= (\cos(\theta) + \sin(\theta) e_{jk}) \phi_A + \phi_C \quad (44)$$

The energy-density of the transformed wave function has remained unchanged (as  $\cos^2(\theta) + \sin^2(\theta) = 1$ ).

An arbitrary rotation about the direction of motion is applied by substituting  $e_{jk}$  in  $R$  for  $e_{ij}$  such that  $R = \exp [e_{ij} (\theta/2)]$ .  $\phi_A$  and  $\phi_C$  for  $R = \exp [e_{ij} (\frac{\theta}{2})]$  are: -

$$\phi_A = \phi_{circ} = (e_{0i} + e_{ki}) \cos(kx_k - \omega\hat{t}) + (e_{0j} - e_{jk}) \sin(kx_k - \omega\hat{t}) \phi_C = 0 \quad (45)$$

Therefore the transformed function is: -

$$\phi'_{circ} = R^2 \phi_{circ} \quad (46)$$

$$\phi'_{circ} = (\cos(\theta) + e_{ij} \sin(\theta)) ((e_{0i} + e_{ki}) \cos(kx_k - \omega\hat{t}) + (e_{0j} - e_{jk}) \sin(kx_k - \omega\hat{t})) \quad (47)$$

The wave has been rotated about the  $\hat{x}_k$  axis and a relative phase change of  $\theta$  has been introduced by this transformation. The energy-density of the wave, once again, has remained unchanged. The four derivative of this wave function remains zero ( $dF = 0$ ).

The behaviour of a wave function under simultaneous compound rotations will now be examined. The aim here, is to illustrate the fact that the order in which rotations are applied affect the overall outcome. Two rotors  $R_1$  and  $R_2$  are defined; each of which correspond to an arbitrary rotation in a different plane

(e.g.  $\hat{x}_i - \hat{x}_j$  and  $\hat{x}_j - \hat{x}_k$ ): -

$$R_1 = \exp \left[ e_{jk} \frac{\theta}{2} \right] \quad (48)$$

$$R_2 = \exp \left[ e_{ij} \frac{\eta}{2} \right] \quad (49)$$

A general expression for the compound rotations of the wave function  $\Psi$  is: -

$$\Psi' = R_1 R_2 \Psi R_2^{-1} R_1^{-1} \quad (50)$$

which will now be expanded in terms of elements which commute with the rotor functions as follows: -

$$\begin{aligned} \Psi' = & \Psi_{(CR_1)(CR_2)} + R_1^2 \Psi_{(AR_1)(CR_2)} + \left( R_2^2 \Psi_{(CR_1)(AR_2)} \right)_{CR_1} + R_1^2 \left( R_2^2 \Psi_{(CR_1)(AR_2)} \right)_{AR_1} \\ & + \left( R_2^2 \Psi_{(AR_1)(AR_2)} \right)_{CR_1} + R_1^2 \left( R_2^2 \Psi_{(AR_1)(AR_2)} \right)_{AR_1} \end{aligned} \quad (51)$$

Each instance of  $\Psi$  on the right hand side of the equation has two subscripted indices which denote their commutation relations with both the rotors being applied. E.g.  $\Psi_{AR_1}$  anti-commutes with the rotor  $R_1$  and  $\Psi_{(AR_1)(AR_2)}$  anti commutes with both  $R_1$  and  $R_2$ . In cases where part of a term's commutation relation may have changed after a rotation has been applied, this partition has been encapsulated in brackets and given a single subscript to indicate the commutation relationships with other rotors.

The ordering of the application of rotations has an impact on the final result. Depending on how  $R_2$  transforms,  $\Psi_{AR_2}$  impacts on what is present for  $R_1$  to transform (i.e. what components anti-commute with  $R_1$  after  $R_2$  has been applied). The exceptions to this are of course the  $\Psi_{(CR_1)(CR_2)}$  term, which is invariant by definition; and the  $R_1^2 \Psi_{(AR_1)(CR_2)}$  term which is invariant under an  $R_2$  transform but not an  $R_1$  transform.

Reversing the ordering of the application of the rotations yields a different multivector transformed expression: -

$$\begin{aligned} \Psi' = & \Psi_{(CR_2)(CR_1)} + R_2^2 \Psi_{(AR_2)(CR_1)} + (R_1^2 \Psi_{(CR_2)(AR_1)})_{CR_2} + R_2^2 (R_1^2 \Psi_{(CR_2)(AR_1)})_{AR_2} \\ & + (R_1^2 \Psi_{(AR_2)(AR_1)})_{CR_2} + R_2^2 (R_1^2 \Psi_{(AR_2)(AR_1)})_{AR_2} \end{aligned} \quad (52)$$

The only term which is guaranteed to be the same in both transformations is the invariant  $(R_1^2 \Psi_{(AR_1)(CR_2)})$ . If  $R_1 = R_2$  then the transformations would be equivalent; in any other case the ordering of rotations will produce a different outcome. Note that even if the final orientation of the multivector  $\Psi$  is the same for different  $R_1$  &  $R_2$ , the geometric phase of the system will vary[96].

## 5.5 Lorentz Invariance

Lorentz invariance will be examined in this section. The essence of Lorentz invariance is that the form of a physical law must be invariant under a Lorentz transformation and that the laws of physics must have no preferred inertial frame of reference. There are also certain quantities which remain unchanged when a Lorentz transformation is applied (i.e they are the same in all inertial frames). Such quantities are said to be *Lorentz invariants*. Known Lorentz invariant quantities will be obtained here and compared with the work of Morse *et al*[78]. The properties of the vector set (t,x,y,z) under a single transformation will be examined here and used to find quantities which are Lorentz invariant. A general rule for finding all invariant quantities for all multivectors under arbitrary transformation will then be derived.

For any space-like vector (x,y or z), a single boost  $\gamma$ , parallel to that vector yields a transformed vector  $\hat{k}'$ , which is always of the form: -

$$\hat{k}' = e_i(\gamma + \beta\gamma e_{0i}) \quad (53)$$

In Eq. (53),  $i$  represents the direction of the  $\hat{k}$  axis, and N is an arbitrary scalar factor. The square of the transformed vector has the form: -

$$\hat{k}'^2 = -(\gamma^2 - \beta^2\gamma^2) \quad (54)$$

Eq. (54) is a scalar. The anti-symmetric portions of the products cancel each other as a result of the non-commutative properties of the algebra. Similarly for the time axis: -

$$\hat{t}' = e_0(\gamma + \beta\gamma e_{0i}) \quad (55)$$

$$\hat{t}'^2 = (\gamma^2 - \beta^2\gamma^2) \quad (56)$$

The factor  $\gamma^2 - \beta^2\gamma^2$  is equal to 1.

$$\gamma^2 - \beta^2\gamma^2 = \left[ \frac{1}{1 - (v/c)^2} - \frac{(v/c)^2}{1 - (v/c)^2} \right] = \frac{1 - (v/c)^2}{1 - (v/c)^2} = 1 \quad (57)$$

Therefore the square of any vector basis-element, of the form  $e_i$  or  $e_0$ , which has been subjected to a single Lorentz transformation, is always equivalent to the square of the vector in the unprimed frame.

A more general rule for finding Lorentz invariants will now be sought. This rule will hold for any number of transformations and includes all multivector elements. A general expression for the square of an arbitrarily transformed multivector  $\bar{M}$  is given in Eq. (58).

$$(\bar{M}')^2 = (\kappa_1\kappa_2 \dots \kappa_N \bar{M} \kappa_N^{-1} \dots \kappa_2^{-1}\kappa_1^{-1}) (\kappa_1\kappa_2 \dots \kappa_N \bar{M} \kappa_N^{-1} \dots \kappa_2^{-1}\kappa_1^{-1}) \quad (58)$$

Note that in these equations, any  $\kappa_n$  and its corresponding  $\kappa_n^{-1}$  are interchangeable with an arbitrary rotor function  $R$  and  $R^{-1}$  respectively. Eq. (58) can be reduced to: -

$$(\bar{M}')^2 = \kappa_1\kappa_2 \dots \kappa_N \bar{M} \bar{M} \kappa_N^{-1} \dots \kappa_2^{-1}\kappa_1^{-1} \quad (\text{for all } N) \quad (59)$$

$$(\bar{M}')^2 = \kappa_1\kappa_2 \dots \kappa_N \bar{M}^2 \kappa_N^{-1} \dots \kappa_2^{-1}\kappa_1^{-1} \quad (\text{for all } N) \quad (60)$$

In Section 5.2, "The Lorentz Transformation" it was shown that scalars and the pseudoscalar ( $e_{0123}$ ) are invariant under all transformations in this algebra. Therefore, a general statement regarding Lorentz invariants can be made: *if*

*the square of a multivector produces any scalar and/or pseudoscalar components, then those components of this squared quantity are the same in all inertial frames.*

Note that there may also be other quantities which are invariant under particular (special case) transformations. Table.5.1 in Section 5.2, “The Lorentz Transformation” can be used to determine these special cases.

A common example from the literature[78, 97, 19] can be used to demonstrate this; the square of the interval  $S$ , where: -

$$S = \hat{t} - \hat{x} - \hat{y} - \hat{z} \ ; \ S' = \hat{t}' - \hat{x}' - \hat{y}' - \hat{z}' \quad (61)$$

the square of  $S$  is known to be a Lorentz invariant[78, 97, 19]. i.e.  $S^2 = (S')^2$ . In this case  $S$  is a four-vector and the square of  $S$  and  $S'$  is a scalar. It has been shown from Eq. (58), (59) & (60) that the expression can be reduced to the form of a transformation of the squared quantity, and the scalar is known to be invariant under all Lorentz transformations in this algebra. Therefore, the transformation that accounts for the difference between  $S$  and  $S'$  can be discarded from the squared expression, and thus the two quantities are equivalent. Notice that, as discussed in Chapter 2, there is no need to introduce the contravariant vector for orthogonal spaces in order to determine invariant quantities.

## 5.6 Energy, Momentum & Electromagnetic Mass

### 5.6.1 Introduction

In this thesis so far, the electromagnetic field equations have been presented and shown to be covariant for the treatment of electromagnetic waves. The treatment of the energy-momentum in terms of electromagnetism is, however, not so straightforward. Although the primary focus of this section is to determine if the electromagnetic momentum and massive properties of transverse wave behave covariantly in this algebra, it is helpful to draw on discussions of the self-field interaction of the electron. The Boyer-Rohrlich controversy[21, 23] provides an interesting discussion on the conflicts between relativistic and non-relativistic electrodynamics. In this section an interpretation of Boyer’s and Rohrlich’s point of view will be presented, their differences will be traced to completely different conceptions of what a relativistic theory should be[89, 90]. The details of this

examination have been presented by Campos *et al*[89] & Moylan[90] and the discussions put forth here follow from these works. The Boyer-Rohrlich controversy will be used to discuss the form of the energy-momentum in the electromagnetic formalism presented in this thesis.

### 5.6.2 The Boyer-Rohrlich Controversy

The account of the Boyer-Rohrlich controversy begins with the Abraham-Lorentz model of the electron[98] which leads to a value of  $\frac{4}{3}\bar{E}/c^2$  for the electromagnetic mass[21, 23]. Rohrlich[99, 22] discusses the properties of the energy-momentum tensor and presents a treatment applicable to macroscopic systems. In this treatment Rohrlich introduces a symmetric energy-momentum tensor as a four-vector and shows that it transforms covariantly\*

Boyer[21], argues that no change in the classical Abraham-Lorentz model of the electron is required for a proper relativistic formalism. Boyer presents “*a simple example which clarifies the traditional Lorentz transformation difficulties of the classical electron model*” and “*which again confirms the standard definition of electromagnetic momentum density as the correct choice for the classical electromagnetic theory*”. Boyer continues, highlighting statements by Poincaré expressing that only the *total* energy and momentum can be expected to satisfy covariant behaviour when transformed between inertial frames and that only the transformed fields may be utilised to derive the energy-momentum. Boyer also shows that Rohrlich’s treatment requires the definition of a velocity with respect to some preferred inertial frame.

Rohrlich’s[23] reply demonstrates that the external “stabilising forces” present in Boyer’s argument prohibit the closed-system description. Furthermore, Rohrlich shows that the choice of integral is independent of the surface chosen for the integral. Despite all this, Rohrlich’s formalism is a point-like description of the electron in which the infinite self-energy problems associated with the Coulomb description have been solved by introducing a quantum mechanical description for classical electromagnetism; effectively enforcing a  $E = mc^2$  condition.

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\*Note that Rohrlich’s metric tensor  $g_{\mu\nu}$  is the anti-Lorentzian form.



### 5.6.3 Energy-Momentum & The Space-Time Split

The Abraham-Lorentz energy-momentum “tensor”, for a Lorentz gauge condition, may be described in  $\mathcal{C}\ell_{1,3}$  by  $\Phi$  as: -

$$\Phi = \frac{1}{2} F F^\dagger = U - \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} \vec{S} \quad (62)$$

In Eq. (62) the energy-momentum does not appear in four-vector form, although it does have four components. The energy density  $U$  appears with scalar form and the momentum with the dimensions of a spacetime-like bivector. This energy-momentum quantity does not transform covariantly under Lorentz transformation as energy is an invariant and the momentum is invariant under parallel boosts (Refer to Table. 5.1).

Following from Boyer’s point, that the total energy-momentum tensor is a derived quantity in terms of transformed fields leads to Eq. (63)

$$\Phi = \frac{1}{2} \kappa F \kappa^{-1} (\kappa F \kappa^{-1})^\dagger = U' - \vec{S}' \quad (63)$$

Eq. (63) demonstrates that if the transformed fields are taken then the derived quantities for energy and momentum vary covariantly.

There is an a “algebraic-device” used by Hestenes[34] to adjust the scalar-bivector form of the energy momentum tensor. This device is known as the spacetime-split and involves the convolution of  $\Phi$  with a future-pointing time-like vector unit  $e_0$ . This method has the effect of transforming the energy-momentum tensor into a four vector which is relativistically covariant.

### 5.6.4 Discussion & Summary

Hestenes’s physical justification for applying this space-time split operator to coordinate systems is that a unique time is then assigned to every event in spacetime[34]. Further justification is given by Hestenes when the operator forms part of a world-time derivative  $\frac{d}{d\tau} e_0$ . Despite producing a covariant quantity, application of this derivative gives force terms and therefore the result is no longer representative of pure energy-momentum.

The root of the Boyer-Rohrlich controversy lies in the non-covariance of the definitions of the energy-momentum in the Abraham-Lorentz model of the electron[90]. An attempt to remedy this 4/3 problem was put forth by Poincaré by postulating that the that an electron's total four momentum should include terms which account for the stability of the electron. However the origin of these forces was not satisfactorily explained. A theory which view the electron as a structure-less entity was forwarded by Fermi[100] and elaborated by Rohrlich. However, as Rohrlich points out, this treatment does not take into account the cohesive stresses and concludes that "we seem to have a relativistic but unstable electron".

Although the energy-momentum associated with the photon does not require cohesive forces, the form of the energy momentum in the fields requires covariant treatment. It would appear that from the point of view of describing relativistic electromagnetism, the Boyer standpoint, that only the transformed fields may be used to derive the energy-momentum, appears to hold true in the formalism presented in this thesis. The Rohrlich standpoint requires the redefinition of the field energy equations. Whilst this is perfectly valid for electrons in a quantum theory, it does not hold true for relativistic electromagnetic fields. This may be the correct approach for drawing a comparison between classical theory and quantum electrodynamics, but the relationship between classical and quantum electrodynamics remains open[89].

## 5.7 Discussion

The differences between the transformations presented in this chapter, and those developed in Morse et al[78] are that commuting and non-commuting objects behave differently in the transformation. For Morse *et al* there are no non-commuting objects and therefore no distinction is made. Most notable is the way in which scalar objects behave under transformation. Due to the commutative properties of the scalar (i.e it commutes with everything), it is the same in all inertial frames for the transformations developed in this chapter.

In this chapter the fields have been shown to transform covariantly. The algebra has been shown to be capable of transparently selecting the appropriate fields for transformation. The energy in the field ( $\vec{E}^2 + \vec{B}^2$ ) has been shown, by the

using of a modified Doppler shift equation, to be covariant only when the fields represent densities. In order for the system as a whole to conserve energy under a Lorentz boost the total energy of a transformed wave equation is, by necessity, defined here as the volume integral of the energy density  $U = \frac{1}{2} [\vec{E}^2 + \vec{B}^2]$ . The shape and size of this volume is irrelevant as it divides itself out of the final equations.

The usual relativistic notation (tensor algebra) requires great care in order to obtain covariance of fields[34]. This tedious manual process is eliminated in the spacetime algebra  $\mathcal{Cl}_{1,3}$  as the basis elements themselves take over this function. There are, however, still issues surrounding the covariance of the energy-momentum density. In this chapter the behaviour of this quantity has been shown to behave in the same way as other formalisms. The algebraic devices of Hestenes[34] has been shown to be capable of correctly for this anomaly and restores covariance. Although this device lacks a physical justification. The vector nature of the four derivative gives rise to a system which acts like a measurement made in a different inertial frame. This demonstrates that the derivative behaves covariantly under transformation. Although a connection is made with the usual covariant derivative, two are not entirely equivalent as here an orthonormal basis  $\{e_\mu\}$  is assumed. Usually, covariant derivative contains positional information with respect to a fixed point of observation. This information can be used to calculate the retarded potentials[30] and other observables within the light cone[53].

A correspondence has been made between the invariants presented in this section and those from other formalisms[78]. The mathematics of how these quantities are shown to be invariant is obtained differs. Many formalisms[30] require the introduction of a contravariant vector in order to obtain a Lorentz invariant scalar product. The product of a co-contravariant pair yields a Lorentz invariant. i.e.  $\Phi_\mu \cdot \Phi^\mu = \Phi^2$ . Which is the same result as raising a four vector to power of two in the spacetime algebra  $\mathcal{Cl}_{1,3}$ . Although the possibility for the introduction of co-contravariant notation is left open, it is not required for the calculation of invariant lengths and intervals in this algebra. The algebra used by Morse *et al* (ict) does not require the introduction of a co-contravariant system there are, however, problems associated with their particular choice of algebra which arise in different areas of electromagnetism and gravitation[101].

## 5.8 Chapter Summary

In this section the Lorentz transform has been presented in two ways; firstly as seen in the work of Morse *et al*[78]; and secondly by the introduction of the transformation seen in the work of Hestenes *et al*[34, 38]. The latter of these has been investigated and the work that followed was aimed at ascertaining its properties and any potential problems with this approach. This investigation has provided an extensive insight into the properties of these transformations in the spacetime algebra  $\mathcal{Cl}_{1,3}$ .

The software developed in Chapter 3, has enabled a systematic approach to be used in the investigation of the behaviour of basis elements and wave functions in the spacetime algebra. The scripting system has enabled the construction of symmetry tables which show how each basis element behaves under Lorentz rotations and boosts; enabling the search for basis elements which transform in a particular way to be performed quickly and easily. These tables will have applications beyond the scope of this thesis.

The investigation has shown that there are symmetries between whether a quantity is invariant under a transformation (boost) with the number of indices of the basis elements (odd or even) and the perpendicularity of the multi-vector with respect to the direction of the boost. There are certain quantities ( $e_0$  &  $e_{123}$ ) which vary with a boost regardless of the direction. There are also other quantities ( $\mathbb{1}$  &  $e_{0123}$ ) which are always invariant.

Additionally, there is a different symmetry for rotations (See Table. 5.1) compared to that of the boosts. In the case of rotations there is no dependence in the number of indices (Odd or Even) of the element under rotation on whether or not the element is rotated; unlike the case for the boosts. This leads to the conclusion that an element is only rotated by this transformation function if it is perpendicular to the axis of rotation.

The behaviour of field wave equations from Chapter. 4 under Lorentz transformation have been investigated. The field elements have been shown to transform covariantly. The wave equations have been shown not to conserve energy unless the field elements themselves represent densities. Comparing the energies of the transformed fields over a volume integral of energy density, with the value from a modified Doppler shift equation, the field equations have been shown only to

obey an energy conservation equation when representing densities.

In this section an investigation of proper quantities[78, 19, 85, 86, 87] has been conducted. The investigation also examined the behaviour of the differential operator under transformation. The behaviour of the differential operator under transformation shows that lengths and time scales are covariant, and it is therefore identified as the “proper” derivative. This derivative has been used to highlight that a properly relativistic solution of the Homogeneous Maxwell’s equations is only a valid one in one inertial frame. The same system, when observed from any other frame, exhibits both fields and current. An investigation into the invariance of higher order derivatives has shown that the general rule for Nth order derivatives is that even ordered derivatives are invariant and that the odd ordered derivatives are covariant.

Methods for finding invariant quantities have been given in the chapter. If the square of any multivector quantity has a scalar or pseudoscalar component, then those quantities are Lorentz invariants. The salient results from the investigation into Lorentz transformations are that scalars in this algebra are invariant under inertial frame transformations and that field quantities in the algebra are covariant. This has been shown in Tables 5.1 & 5.2 and is a consequence of the commutation relations of the algebra.

# Chapter 6

## Conclusions

In this thesis the utility of the Spacetime algebra  $\mathcal{Cl}_{1,3}$  in representing relativistic electromagnetism has been investigated. The algebra presented has been shown to be a real, Dirac, Minkowskian and Lorentzian algebra. A sub-group of the algebra has been identified as isomorphic with the Quaternion group. The Lorentz group has also been identified within the algebra.

In this thesis some simple as well as some brute force methods for determining roots of basis elements have been presented. An exhaustive computer search algorithm, along with a new test procedure, have been developed to search for these roots. The results presented show that unique roots of non-scalar basis elements must contain an even number of linearly independent, non-repeating terms. The square of the sums of linearly independent roots of the  $e_{\ominus}$  elements has been shown to be equal to the sum of the squares. It has also been shown that unique roots of  $e_{\oplus}$  elements, those which square to unity, are not found in roots containing two or six terms. The requirement for roots of members of  $e_{\oplus}$  to have exactly four terms has been demonstrated. No unique roots are found where the number of terms exceeds six; leading to the conclusion that all unique roots have been found by this point.

The differences between the basis element properties of the four dimensional spacetime algebra (+ - - -) and that of the three dimensional Euclidean algebra (+ + +) have been investigated in this thesis. The comparison of the two systems was conducted by computer program and checked and confirmed manually. The results of this comparison were listed in tables. The analysis of this data

showed that for the normal interior product, the projection is isomorphic. The handedness of rotations, however, are not equivalent for the four-space Lorentz metric and the three-space Euclidean metric. Rotations are right handed in  $\mathcal{Cl}_{1,3}$  and are consistently left-handed in  $\mathcal{Cl}_{3,0}$ . By altering the sign of the wedge,  $\wedge$  product for this particular three-space metric, the handedness is reversed. In conclusion, the mapping given in this thesis are not isomorphic unless the definition of the full product for the three-space algebra is changed to take into account the handedness of rotations in the projection. Unless this alteration is made, rotations in the two different Clifford algebras will differ, as will the geometric phase. Altering the three-space product in this way yields a new algebra which is not strictly a Clifford algebra.

The development of software capable of manipulating the algebra has been justified and presented in this thesis. Some of the various uses to which this software may be applied have been demonstrated. The algebraic software presented in this has fulfilled the specifications in Chapter 3 and although other software now exists which is capable of performing many of the required operations, they were not available when development commenced. This software has enabled the examination of very large phase space problems such as the exhaustive root search and the examination of symmetries of the Lorentz Boosts & Lorentz Rotations.

A simple algorithm which may be used for calculating the product of basis elements in Clifford algebras  $\mathcal{Cl}_{n,m}$  and the Dirac  $\gamma$ -matrix algebra has been presented in this thesis. The algorithm has been shown to be reliable and adaptable to any required metric. The reliability is derived from the sorting mechanism utilised for ordering terms prior to the cancellation procedure. The process of cancellation requires that a choice of metric be made. A system which yields the Lorentz metric (+ - - -) has been implemented here. This algorithm is a simple, fast and reliable means of performing the mechanical calculation of products in this algebra.

The electromagnetic field equations have been derived in the spacetime algebra  $\mathcal{Cl}_{1,3}$ . This derivation has shown that for the Lorentz gauge condition, all the equations appear at once, with all the correct signs and with no requirement to introduce a separate dual field strength tensor to obtain the homogeneous equations. The introduction of the dual field strength tensor has been demonstrated and has been shown to be required in the usual relativistic notation[30]. As a

result the field equations presented in this thesis appear in a particularly compact and elegant form.

The transverse wave functions solutions to the field equations presented in this thesis have illustrated methods and techniques for replacing waves containing the complex imaginary  $\mathbf{i}$  with a real system. These methods have been used to represent the electric and magnetic fields solutions to Maxwell's equations in a single purely real function. Although more complicated than the conventional approach, this method does have the advantage of being real everywhere.

A new set of wave functions, which have no mathematical analogy in either the quaternion algebra or relativistic notation have been introduced and examined in this thesis. These functions have been shown to be equivalent to the standard wave functions for cases where the propagation operator  $\phi_{prop}$  and the fields match such that  $\vec{E} \perp \vec{B} \perp \vec{S}$  where  $\vec{S}$  is the propagation direction. A further condition that the electric and magnetic fields must be of the same magnitude has been demonstrated. As a result these new wave functions only propagate a limited set of real and physical fields. Problems with finding a vector potential description using this new format for the wave functions have been discussed. The conclusion from the analysis is that the format of the function prohibits derivation from a four potential in this formalism.

The gauge degree of freedom has been introduced in this thesis in terms of the dynamics of the four-potential. A demonstration of how the algebra may be utilised to extend and simplify the field equations in terms of a homogeneous wave equation of the four potential, which still supports the introduction of charge density through the gauge has been presented here. General gauge transformations have been introduced and the Lorentz and Coulomb gauge conditions presented in this algebra. By defining  $d^2A = 0$  and  $dL = -J$  the possibility to set the charge, through the introduction of a *scalar field* together with the gauge has been examined. It has been shown that the the gauge cannot be global to do so, but instead must change close to the position of the charge. The consequences of deviating from a global gauge condition have been examined and the effects on local charge conservation have been examined. The destruction of gauge invariance in this situation has been shown.

The properties of transverse electromagnetic wave functions under general Lorentz transformation have been examined. The form of the transform in this



algebra has been presented as an extension of the Galilean transform. Using the software the properties of arbitrary quantities under transformation have been examined and shown to transform covariantly. The transformation of the differential operator has been shown to correspond to a covariant or “proper” derivative. The Laplacian operator  $\nabla^2$  has been presented and shown that it and its four dimensional counterpart  $\square$  are invariant operators. Furthermore, a general mathematical treatment of differentials has led to the a proof that all even powers of the four vector derivative are invariants in this algebra and that all odd ordered derivatives are covariant. Invariants have been examined and it has been shown that scalars are true Lorentz scalars in this algebra.

An examination of electromagnetic energy-momentum in the presented formalism has shown that only the transformed fields may be used to derive this quantity. The lack of covariance of the energy-momentum terms has been compared to the work of Boyer[21] and Rohrlich[23]. This comparison has shown that the standpoint of Boyer is preferable when examining relativistic electromagnetism in this algebra and formalism.

Finally, from the research presented in this thesis, the conclusion is put forth that the spacetime algebra  $\mathcal{Cl}_{1,3}$  is capable of representing electromagnetic field equations and their solutions without the introduction of the complex number  $i$  or a separate dual field strength tensor. With the single exception of the new wave functions there is, however, no operation or calculation which could not be performed in the usual relativistic notation provided sufficient care and attention is taken. The new wave functions presented have no mathematical analogy in either the usual relativistic notation or the algebra of forms[4].

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# Appendix A

## Field Equations - Relativistic Notation

In this appendix we will show explicitly the expansion of the product  $d_\mu F^{\mu\nu} = J^\nu$ . The field strength tensor  $F^{\mu\nu}$  is given in J.D Jackson's "Classical Electrodynamics" as: -

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad (1)$$

The four derivative, denoted by  $d_\mu$  is: -

$$d_\mu = \left[ \frac{d}{\partial t} \quad \frac{d}{\partial x} \quad \frac{d}{\partial y} \quad \frac{d}{\partial z} \right] \quad (2)$$

and it's contravariant partner  $d^\mu$ : -

$$d^\mu = \left[ \frac{d}{\partial t} \quad -\frac{d}{\partial x} \quad -\frac{d}{\partial y} \quad -\frac{d}{\partial z} \right] \quad (3)$$

The full product,  $d_\mu F^{\mu\nu}$  is given by the product of the vector  $d_\mu$  with the matrix  $F^{\mu\nu}$ : -

$$d_\mu F^{\mu\nu} = \begin{bmatrix} \frac{d}{dt} & \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} = J^\nu \quad (4)$$

which gives: -

$$d_\mu F^{\mu\nu} = \begin{bmatrix} 0 + \frac{c}{dx} E_x + \frac{d}{dy} E_y + \frac{d}{dz} E_z & -\frac{d}{dt} E_x + \frac{d}{dy} B_z - \frac{d}{dz} B_y & -\frac{d}{dt} E_y - \frac{d}{dx} B_z + \frac{d}{dz} B_x \\ -\frac{d}{dt} E_z + \frac{d}{dx} B_y - \frac{d}{dy} B_x & & \end{bmatrix} \quad (5)$$

# Appendix B

## Power Series

In this appendix the properties of various power series in the spacetime algebra will be examined and a connection to some well known functions will be made. This appendix is provided for reference only. The intention is to show how the expansion of the exponential power series can be broken down in terms of sines and cosines, hyperbolic or sinusoidal. This is presented as a useful point of reference for the reader.

The exponential power series is: -

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \quad (all\ x) \quad (1)$$

The product of x with an object which squares to -1 (e.g.  $i$ ) transforms this series to: -

$$e^x = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} \dots + \frac{(ix)^n}{n!} \quad (all\ x) \quad (2)$$

The “complex” and real parts of this series can now be readily identified with the sinusoidal power series: -

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \dots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} \quad (all\ x) \quad (3)$$

and the cosine series: -

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{2n!} \quad (all\ x) \quad (4)$$

Therefore Euler's theorem has been demonstrated: -

$$\exp[ix] = \cos(x) + i\sin(x) \quad (5)$$

The properties of the complex number  $i$  which make this separation possible are that it squares to  $-1$  and the linear independence of the real and complex number systems. Therefore, the same translation can be applied to a multivector which squares to  $-1$ .

# Appendix C

## Using the Algebraic Software

### C.1 Initialising

The algebraic software as described in Chapter 3, is implemented as a Python module. This module is initialised from the python command line using the command: -

```
from algebra import *
```

this will load the algebraic software modules and report any optimisation libraries which have been loaded.

### C.2 Grade()

A Grade instance has the Lorentz metric (+ - - -) and is created using the following command: -

```
ei = Grade('i')
```

or

```
e0 = Grade('0')
```

where the parameter passed corresponds to a string which is passed directly to the reduction algorithm. Note that by default there are several instances of grade classes automatically created when the algebra module is loaded. This includes

an object of the form  $e\#$ , where  $\#$  represents the sub element formations for for each of the sixteen basis elements. e.g.  $e1 = e_1$ ,  $e31 = e_{31}$  etc.

### C.3 AGrade()

An AGrade instance has the anti-Lorentz metric  $(- + + +)$  and is created using the following command: -

```
hi = AGrade('i')
```

or

```
h0 = AGrade('0')
```

where the parameter passed corresponds to a string which is passed directly to the reduction algorithm. Note that by default there are several instances of grade classes automatically created when the algebra module is loaded. This includes an object of the form  $h\#$ , where  $\#$  represents the sub element formations for for each of the sixteen basis elements. e.g.  $h1 = h_1$ ,  $h31 = h_{31}$  etc.

### C.4 Variable()

A variable instance is created using the following command: -

```
x = Variable('x')
```

where is replaceable with any letter of choice. Note, however, that the following two variable instances are equivalent: -

```
x = Variable('z')
y = Variable('z')
```

### C.5 .eval()

`.eval()` is used by passing dictionary of values to an expression's `.eval()` method.

An example of such a Python dictionary might be: -

```
{ 'x':3.14 , 'y':2*3.14 , 'e':1.6E-19 }
```

which is passed to an expression as: -

```
expr_of_x_y_and_e.eval({ 'x':3.14 , 'y':2*3.14 , 'e':1.6E-19 })
```

Note that not all parameters need to be used, and that parameters may be used more than once during an evaluation cycle. In the case where a parameter is not provided for an expression, no error occurs; the variable will remain where it appeared before.

## C.6 .seval()

.seval() is used by passing a Grade() instance to an expression's .seval() method.

For example: -

```
(e0*e1).seval(e0)
```

will return zero. However: -

```
(7*e0+e1).seval(e0)
```

Will return "7\*e0". By utilising this facility, one is able to separate out the sixteen basis elements from the results of a calculation and plot them individually.

## C.7 Differential Operator

The differential operator is defined in the code as a summation lambda operation:

```
D = lambda A : A._D_(t)*e0 - A._D_(x)*e1 \
- A._D_(y)*e2 - A._D_(z)*e3
```

which can be utilised on any instance of an Expression or Term class (and all other objects defined within the software) like: -

```
D(Objectname)
```

which returns the differentiated object.



## C.8 Wave()

An instance of a Wave() class may be created by: -

```
A0=1
k = 1
w = 1
t = Variable('t')
z = Variable('z')
A = A0*Wave(i*(k*z-w*t))
```

Here the Wave() class constructor creates an instance of an exponential wave function, with arguments " $i * (k * z - w * t)$ ". Similarly for a purely real wave: -

```
A0=1
k = 1
w = 1
t = Variable('t')
z = Variable('z')
A = A0*Wave(e12*(k*z-w*t))
```

An instance of a Wave() class may be expanded in terms of its constituent power series functions with the ".eval()" function: -

```
A.eval()
```

## C.9 Example Script

```

# Load the algebraic software module into python
from algebra import *

# generate the even subalgebra relations
es = [1, e01, e02, e03, e23, e31, e12, e0123]
gs = [1, g1, g2, g3, g23, g31, g12, g123]

header = """<table>"""
footer = """</table>"""
output = header

# start a loop over all the 3d objects

for i in range(0, len(gs)):
    output = output + """<tr><td colspan="4"><hr /></td></tr>"""
    # for each and every 3d object, multiply by
    # every other 3d object. and show the results
    for j in range(0, len(es)):
        # Do the multiplications
        A = es[i]*es[j]
        B = gs[i]*gs[j]
        # check if the results match
        equiv = (A == B) + (e0*A == B)
        # output the data to the table
        output = output + '<tr><td>' + str(es[i]) + \
            '</td><td>' + str(es[j]) + '</td><td>' + str(A) + \
            '</td><td>&nbsp;</td><td>' + str(gs[i]) + \
            '</td><td>' + str(gs[j]) + '</td><td>' + str(B) + \
            '</td><td>' + str(equiv) + '</td></tr>'
    output = output + footer
# write the output to a file
f = open("eventothree.html", "w")

```

```
f.write(output)
f.close()
```

# Appendix D

## Notation Cheat Sheet

### D.1 Products

The inner or interior product acts on row & column vectors only\*: -

$$\vec{A} \cdot \vec{B} = A_i B_i, \quad i = \{1 \dots 3\} \quad (1)$$

Similarly for the cross product (again acting on row and column vectors only) : -

$$\vec{A} \times \vec{B} = (A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1) \quad (2)$$

The Clifford or Geometric product: -

$$AB = A \cdot B - A \wedge B \quad (3)$$

The symmetric portion of the Clifford product (also uses the dot notation): -

$$A \cdot B = \frac{1}{2} (AB + BA) \quad (4)$$

The anti-symmetric portion of the Clifford product: -

$$A \wedge B = \frac{1}{2} (AB - BA) \quad (5)$$

Heaviside-Gibbs Hybrid (where the row-vector multiplied by the column vector

---

\*Note no basis elements are implied in row vectors.

implies an inner product):-

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} = (e_1 A_1 + e_2 A_2 + e_3 A_3) \quad (6)$$

A four vector: -

$$e_\mu A_\mu = e_0 A_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} \quad (7)$$

Example of where the inner product is anti-symmetric. Let: -

$$A = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A}, B = \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} \vec{B} \quad (8)$$

$$AB = A \cdot B + A \wedge B = e_0 \vec{A} \cdot \vec{B} - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (\vec{A} \times \vec{B}) \quad (9)$$

$$BA = B \cdot A + B \wedge A = -e_0 \vec{B} \cdot \vec{A} - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (\vec{A} \times \vec{B}) \quad (10)$$

giving: -

$$A \cdot B = \frac{1}{2} (AB + BA) = - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} (\vec{A} \times \vec{B}) \quad (11)$$

$$A \wedge B = \frac{1}{2} (AB - BA) = e_0 \vec{A} \cdot \vec{B} \quad (12)$$

which is a curious example of the inner product being anti-symmetric and the outer product being symmetric.

## D.2 Quantities

Here are some familiar quantities and their representations.

$$A = e_0 A_0 + \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \vec{A} \quad (13)$$

$$E = e_{0i} \cdot \vec{E} = \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} \vec{E} \quad (14)$$

$$B = e_{ij} \cdot \vec{B} = \begin{pmatrix} e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} \vec{B} \quad (15)$$

$$F = -E - B \quad (16)$$

## D.3 Further Notational Example

A rather strange example. Please pay particular attention to the over-arrows.  
Let: -

$$E = \begin{pmatrix} e_{01} \\ e_{02} \\ e_{03} \end{pmatrix} \vec{E} \quad (17)$$

$$\nabla E = \nabla \cdot E + \nabla \wedge E \quad (18)$$

$$\nabla \cdot E = \frac{1}{2} (\nabla E + E \nabla) = - \begin{pmatrix} e_{023} \\ e_{031} \\ e_{012} \end{pmatrix} \nabla \times \vec{E} \quad (19)$$

$$\nabla \wedge E = \frac{1}{2} (\nabla E - E \nabla) = e_0 \nabla \cdot \vec{E} \quad (20)$$

Which is, perhaps, the opposite separation you might expect!

