The Geometry of Multiplication: Introducing Clifford Algebra

A White Paper for the Curious

Michael Rudolph

Abstract

Clifford algebra is the natural language for describing geometry, rotations, and physics in any number of dimensions. This paper introduces its core ideas starting from familiar concepts (vectors, dot products, and areas) and shows how one elegant rule generates an entire algebraic system. Only basic vector algebra is assumed.

1 Two Ways to Multiply Vectors

You already know two ways to combine vectors:

The dot product takes two vectors and produces a number (scalar):

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

This measures how much the vectors point in the same direction.

The cross product takes two vectors and produces another vector:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$$

This produces a vector perpendicular to both, with magnitude equal to the area of the parallelogram they span.

This situation invites a question: why do we need two separate operations? And why does the cross product only work in three dimensions?

Clifford algebra provides a single, unified product that contains both and works in any number of dimensions.

2 The Geometric Product

The *geometric product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is written simply as $\mathbf{ab}$. It is defined by one fundamental rule:

> **The Fundamental Rule:** For any vector $\mathbf{v}$, the product of $\mathbf{v}$ with itself equals a scalar (number):
>
> $$\mathbf{v}^2 = |\mathbf{v}|^2$$
>
> That is, a vector times itself gives the squared length.

This single rule, combined with the requirement that the product distributes over addition (like ordinary multiplication), generates everything else in the algebra. What follows in later sections are consequences that can be derived from this one rule.

3 What Does "Squared Length" Mean?

Before going further, let's address a natural question: if $10\text{m} \times 10\text{m} = 100\text{m}^2$, is that an area?

It depends on the geometry. The units "m^2" can mean different things:

Area (from perpendicular directions): If you multiply 10m going east by 10m going north, you get the area of a square: 100m^2 of surface.

Squared length (from the same direction): If you multiply 10m east by 10m east, you are going along a line both times. The result, 100m^2, is a *squared length*, a different kind of quantity from area.

The difference is geometric:

- Perpendicular directions span a *plane*
- Parallel directions span only a *line*

3.1 What Question Does Squared Length Answer?

Consider the dot product $\mathbf{a} \cdot \mathbf{b}$. It answers: "How much does $\mathbf{a}$ point along $\mathbf{b}$?"

What about $\mathbf{a} \cdot \mathbf{a}$? It answers: "How much does $\mathbf{a}$ point along itself?"

The answer is: completely, with perfect alignment. So you get $|\mathbf{a}|^2$.

This "squared magnitude" appears throughout physics:

- Kinetic energy: $E = \frac{1}{2}mv^2$
- Pythagorean theorem: $c^2 = a^2 + b^2$
- Wave intensity: proportional to amplitude^2
- Field energy density: $\frac{1}{2}(E^2 + B^2)$

Nature repeatedly "bills" us for the *square* of quantities. The rule $\mathbf{v}^2 = |\mathbf{v}|^2$ builds this directly into the algebra.

4 Perpendicular Vectors: Something New

What happens when we multiply *different* vectors? Consider two perpendicular unit vectors α_1 (pointing east) and α_2 (pointing north).

From the fundamental rule:

$$(\alpha_1 + \alpha_2)^2 = |\alpha_1 + \alpha_2|^2 = 2$$

But expanding the left side:

$$(\alpha_1 + \alpha_2)^2 = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2\alpha_1 + \alpha_2^2 = 1 + \alpha_1\alpha_2 + \alpha_2\alpha_1 + 1$$

For these to be equal:

$$\alpha_1\alpha_2 + \alpha_2\alpha_1 = 0$$

Therefore:

$$\alpha_1\alpha_2 = -\alpha_2\alpha_1$$

Perpendicular vectors *anticommute*: their product changes sign when you swap the order. This property follows inevitably from the fundamental rule.

4.1 What IS $\alpha_1\alpha_2$?

The product $\alpha_1\alpha_2$ is a new kind of object: a **bivector**.

Geometrically, it represents an *oriented plane*, the plane spanned by α_1 and α_2, with a sense of rotation (counterclockwise if you go from α_1 toward α_2).

Operation	Result
Same direction $\times$ same direction	Scalar (squared length)
Perpendicular $\times$ perpendicular	Bivector (oriented plane)

This is exactly your intuition about area, now built into the algebra.

5 The Full Product: Scalar + Bivector

For two general vectors **a** and **b**, the geometric product splits into two parts:

$$\mathbf{ab} = \underbrace{\mathbf{a}\cdot\mathbf{b}}_{\text{scalar}} + \underbrace{\mathbf{a}\wedge\mathbf{b}}_{\text{bivector}}$$

The **dot product** (inner product) extracts the scalar part, measuring how aligned the vectors are.

The **wedge product** (outer product) extracts the bivector part, the oriented plane they span.

For perpendicular vectors, the dot product vanishes (the vectors have zero alignment), leaving only the bivector.

For parallel vectors, the wedge product vanishes (they span only a line), leaving only the scalar.

The geometric product packages both into one operation. This decomposition is a consequence of the fundamental rule applied to general vectors.

6 Building Higher Grades

We can continue multiplying. In 3D, with basis vectors $\alpha_1, \alpha_2, \alpha_3$:

Trivector: The product of all three basis vectors

$$\alpha_1\alpha_2\alpha_3 = \alpha_{123}$$

This represents an oriented *volume*, the unit cube with an orientation (right-handed or left-handed).

6.1 The Pattern

In n dimensions, we get 2^n independent basis elements. The "Count" column below shows how many independent components each grade contributes:

Grade	Name	Geometric Meaning	Count (in 3D)
0	Scalar	Magnitude/point	1
1	Vector	Directed line	3
2	Bivector	Oriented plane	3
3	Trivector	Oriented volume	1
		Total basis elements:	8

The count follows the binomial pattern: $\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}=1+3+3+1=8=2^3$. This reflects the number of ways to choose k basis vectors from n to form a grade-k element.

In 4D (spacetime), we get $2^4=16$ basis elements: 1 scalar, 4 vectors, 6 bivectors, 4 trivectors, and 1 quadrivector.

7 Why Bivectors Are Distinct from Vectors

In 3D, there are 3 independent bivectors: α_{23} (the yz-plane), α_{31} (the zx-plane), and α_{12} (the xy-plane).

Three independent components, just like vectors! This coincidence is why we can represent bivectors using vectors in 3D (the cross product does exactly this).

But this is an accident of 3D. In 4D:

- Vectors: 4 components
- Bivectors: 6 components

They are fundamentally different objects. The cross product works only in 3D, while the wedge product generalizes to any dimension.

7.1 Physical Consequence

The magnetic field **B** is taught as a vector, but it is actually a bivector: it represents circulation in a plane, a different concept from a direction in space. In 3D, we can get away with treating it as a vector perpendicular to that plane: the xy-plane gets associated with the z-direction, and so on.

In 4D spacetime, there are 6 independent bivectors and only 4 vectors. The correspondence breaks down. This is why electromagnetism looks awkward in standard vector notation but becomes elegant in Clifford algebra: the electric and magnetic fields are properly unified as a single bivector field, the *Faraday bivector* F, with 6 independent components (3 for **E**, 3 for **B**).

8 Spacetime: When Squares Can Be Negative

So far, we have assumed $\mathbf{v}^2=|\mathbf{v}|^2>0$. Special relativity demands a richer structure.

In spacetime, with time direction α_0 and space directions $\alpha_1,\alpha_2,\alpha_3$:

$$\alpha_0^2=+1 \quad \text{(time squares to positive)} \tag{1}$$

$$\alpha_1^2=\alpha_2^2=\alpha_3^2=-1 \quad \text{(space squares to negative)} \tag{2}$$

This is the *metric signature*, encoding the fundamental difference between space and time.

8.1 The Interval

Consider a displacement in spacetime:

$$d\mathbf{x}=\alpha_0\,dt+\alpha_1\,dx+\alpha_2\,dy+\alpha_3\,dz$$

Its square is:

$$(d\mathbf{x})^2 = dt^2 - dx^2 - dy^2 - dz^2$$

This is the spacetime *interval*, the quantity all observers agree on, regardless of their motion. The mixed signature (+ for time, − for space) emerges automatically from the algebra.

Why does this work? Expanding $(d\mathbf{x})^2$ using the geometric product:

$$\begin{aligned}(d\mathbf{x})^2 &= (\alpha_0\,dt + \alpha_1\,dx + \alpha_2\,dy + \alpha_3\,dz)^2 &(3)\\ &= \alpha_0^2\,dt^2 + \alpha_1^2\,dx^2 + \alpha_2^2\,dy^2 + \alpha_3^2\,dz^2 + (\text{cross terms}) &(4)\\ &= (+1)dt^2 + (-1)dx^2 + (-1)dy^2 + (-1)dz^2 + 0 &(5)\end{aligned}$$

The cross terms vanish because different basis vectors anticommute: $\alpha_0\alpha_1 + \alpha_1\alpha_0 = 0$. Only the squared terms survive, each weighted by its signature.

8.2 Three Types of Displacement

Type	Condition	Physical Meaning
Timelike	$(d\mathbf{x})^2 > 0$	Slower than light (massive particles)
Spacelike	$(d\mathbf{x})^2 < 0$	Faster than light (forbidden for matter)
Lightlike (null)	$(d\mathbf{x})^2 = 0$	Exactly light speed

For light: $dt^2 = dx^2 + dy^2 + dz^2$, so the interval is zero. Light accumulates zero "length" in spacetime. This is why photons experience zero proper time.

9 Why This Matters

Clifford algebra is elegant mathematics that reveals structure otherwise hidden:

Rotations become simple. A rotation in the xy-plane is generated by the bivector α_{12}. The rotation of vector $\mathbf{v}$ by angle θ is:

$$\mathbf{v} \mapsto e^{-\frac{\theta}{2}\alpha_{12}}\,\mathbf{v}\,e^{+\frac{\theta}{2}\alpha_{12}}$$

The bivector α_{12} specifies *which* plane to rotate in; the angle θ specifies how much. The same formula works for Lorentz boosts in spacetime: replace α_{12} with a spacetime bivector like α_{01}.

Complex numbers and quaternions are included. The subalgebra $\{1, \alpha_{12}\}$ behaves exactly like $\{1, i\}$: notice that $\alpha_{12}^2 = \alpha_1\alpha_2\alpha_1\alpha_2 = -\alpha_1\alpha_1\alpha_2\alpha_2 = -1$, just like $i^2 = -1$. Quaternions appear as $\{1, \alpha_{23}, \alpha_{31}, \alpha_{12}\}$. These familiar number systems are pieces of a larger whole.

Maxwell's equations simplify. All four Maxwell equations can be written as one:

$$\mathrm{d}F = J$$

where $\mathrm{d} = \alpha^\mu \partial_\mu$ is the vector derivative and F is the electromagnetic bivector field. What appears in standard notation as four separate equations with curls and divergences is revealed to be a single geometric statement: the derivative of the field equals the source. The algebra handles the bookkeeping that humans otherwise must do by hand.

10 Summary: One Rule, All of Geometry

Clifford algebra is built from one idea: **vectors square to scalars.**

From this single rule:

- Perpendicular vectors anticommute, generating bivectors (planes)
- Continuing generates trivectors (volumes) and beyond
- The geometric product unifies dot and wedge products
- Rotations and boosts are simple exponentials of bivectors
- Complex numbers and quaternions emerge as subalgebras
- Physics equations become geometric and coordinate-free

The algebra *is* geometry, made algebraic.

11 Further Reading

For those wishing to go deeper:

- Hestenes, D. & Sobczyk, G. (1984). *Clifford Algebra to Geometric Calculus*. Reidel. (The comprehensive reference.)
- Doran, C. & Lasenby, A. (2003). *Geometric Algebra for Physicists*. Cambridge. (Excellent for physics applications.)
- Hestenes, D. (2015). *Space-Time Algebra*, 2nd ed. Birkhäuser. (Focused on spacetime and relativity.)
- Macdonald, A. (2011). *Linear and Geometric Algebra*. CreateSpace. (Gentle undergraduate introduction.)

* * *

The machinery of Clifford algebra may seem elaborate at first. But it is simply the metric, the rule for measuring lengths and angles, taken seriously at the level of multiplication. Once you do, the rest follows inevitably.

The universe, it seems, already knows this algebra. We are simply learning to speak its language.